LIVŠIC THEOREMS FOR NON-COMMUTATIVE GROUPS INCLUDING DIFFEOMORPHISM GROUPS AND RESULTS ON THE EXISTENCE OF CONFORMAL STRUCTURES FOR ANOSOV SYSTEMS

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Abstract. The celebrated Livšic theorem [Liv71] [Liv72a], states that given M a manifold, a Lie group G, a transitive Anosov diffeomorphism f on M and a Hölder function $\eta : M \mapsto G$ whose range is sufficiently close to the identity, it is sufficient for the existence of $\phi : M \mapsto G$ satisfying $\eta(x) = \phi(f(x))\phi(x)^{-1}$ that a condition — obviously necessary — on the cocycle generated by η restricted to periodic orbits is satisfied.

In this paper we present a new proof of the main result. These methods allow us to treat cocycles taking values in the group of diffeomorphisms of a compact manifold. This has applications to rigidity theory.

The localization procedure we develop can be applied to obtain some new results on the existence of conformal structures for Anosov systems.

1. Introduction

The goal of this paper is to give a unified presentation – sometimes involving sharper technical conclusions – of the existence of solutions to coboundary equations over Anosov systems.

We will give precise definitions in Section 3 but we anticipate that the main concern will be whether, given an Anosov diffeomorphism on a manifold M, and function $\eta : M \to G$, where G is a group (either a Lie group or a group of diffeomorphisms), there exists a function $\phi : M \to G$ such that

$$\phi \circ f = \eta \cdot \phi. \tag{1}$$

(We will also discuss the flow case, but we omit a preliminary discussion of it).

In the standard terminology, if we can find a solution to equation (1) then we say that the cocycle generated by η is a coboundary. There are many other variations of this question. For example, instead of taking G to be a Lie group, it is possible to consider G to be a Banach algebra

[BN98] or a bundle map. We will omit other important variations, such as when (M, f) is a subshift. We will mention in Section 6 the situation when ϕ are conformal structures and η is natural map induced by the tangent map. The study of such Livšic theorems in more geometric contexts seems fruitful and will be pursued in further papers.

Cocycles arise naturally in many situations. They are intrinsic to the definitions of special flows and skew products. In the study of dynamical systems, the chain rule indicates that the derivative is a cocycle. The coboundary equation is geometrically natural and hences arises naturally in a number of situations. In particular, (1) appears naturally in the linearization of more complicated equations, for example, it appears in the linearization of conjugacy equations. Hence, cocycle equations are basic tools for the rigidity program [Zim84, GS97, BI02]. Cocycle equations appear also in the study of the asymptotic growth properties of dynamical systems. Diffeomorphism valued cocycles appear when considering the behavior of system relative to its behavior on a factor. The study of (1) with M a shift space, appears naturally in thermodynamic formalism when one tries to decide whether two potentials give rise to the same Gibbs state [Sin72, Bow75].

Note that, when $f^n(p) = p$, the existence of a solution to (1) implies that

$$\eta(f^{n-1}p) \cdots \eta(fp) \cdot \eta(p) = \mathrm{Id}\,.$$

If this necessary condition holds for all periodic points $p \in M$ then we say that the *periodic orbit obstruction* vanishes.

It is natural to ask whether the converse is true. Namely, if given an η such that the periodic orbit obstruction vanishes, whether there is a ϕ solving (1). Another natural question – especially for applications to geometry – is whether the solutions of (1) are regular.

In this paper, we will concentrate in the existence question, but since we will also study the case when G is a group of C^r diffeomorphisms, some regularity considerations will come in.

The question of the existence of solutions to (1) was first studied by Livšic in [Liv71] and [Liv72a], when f is a topologically transitive Anosov system, and in [Bow75] when f was a subshift of finite type. We will refer as *Livšic theorems* to theorems that guarantee the existence of solutions of (1) under the hypothesis that f is a topologically transitive Anosov system or flow.[1] These papers showed that when f is transitive and η is Hölder, then the periodic obstruction is sufficient for

[1] Some references use also the spelling Livshitz. We prefer to maintain the spelling used in the papers by the author and in much of the subsequent literature.

the existence of a Hölder ϕ. (Continuity of η is definitely not enough and there are counterexamples).

There is a considerable literature on Livšic theorems in various contexts. [LS72] shows that for real-valued Hölder cocycles the existence of L^∞ solution to the coboundary equation with a Hölder η implies the existence of a Hölder solution - in the literature this is often called the measurable Livšic theorem. We note that the main difficulty of measurable Livšic theorems is that (1) is not assumed to hold everywhere, but only on a set of full measure. Hence restricting to a periodic orbit – or to the stable manifold of a periodic point do not make sense. The interpretation of the periodic orbit obstruction is far from obvious.

This was extended in the non-commutative case to certain L^p spaces using Sobolev regularity techniques in [dlL01] though interestingly the case of L^1 solutions remains open in the non-commutative case. A version of the measurable Livšic theorem for cocycles taking values in semi-simple Lie groups without any integrability assumptions appears in [NP01] though they need to assume additional bunching conditions on the cocycle. Similarly a version of the measurable Livšic theorem for cocycles taking values in compact Lie groups appears in [PP97]. This was extended, under some additional hypotheses, to the case of cocycles taking values in connected Lie groups [PW01]. Similar results for hyperbolic flows appear in [Wal00b]. That some additional hypotheses are necessary is proved by a counterexample with a cocycle taking values in a solvable Lie group in [Wal00a]. Extensions of the measurable Livšic theorem to more general dynamics appear in [PY99, Dol05]. Analogues for Markov maps or for systems with discontinuities, appear in [NS03, Pol05, BHN05]. The study of the equation for skew-products appears in [PP06, Dol05]. Cohomology equations over higher dimensional actions were studied in [NT03, NT02, FM03]. In general the vanishing of the periodic orbit obstruction can be difficult to verify though in some cases it is implied by spectral data [DG75]. If we can verify that the periodic orbit obstruction vanishes on periodic orbits of period less than T for some T then we may still obtain approximate solutions to the cohomology equation [Kat90].

When M is a quotient of a group and a lattice and f is an automorphism, the equation (1) can be studied using group representation techniques. For example, [Liv72a] considered the case $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and [CEG84] considered $M = PSL(2,\mathbb{R})/\Gamma$ and f a geodesic flow. A more general study of (1) using representation techniques is in [Moo87]. The representation theory methods yield information on the regularity questions also and the first results on regularity appeared in [Liv72a].

The representation theory methods need to assume that f has an algebraic structure, but not that it Anosov [Vee86, FF07]. Of course, the representation theory methods also lead to obstructions. Though the representation theory obstructions must be equivalent to the periodic orbit obstructions, the connection is mysterious.

For the particular case of geodesic flows further geometric information on the solutions of the coboundary equation is obtained in [GK80a] for surfaces, and in [GK80b] for n-dimensional manifolds with a pinching condition. The method of these papers uses harmonic analysis in some directions of the problem and obtains, not only regularity, but also several other geometric properties of the solutions (e.g. that they are polynomial in the angle variables).

The regularity theory for the case when G is commutative and f is any Anosov system appeared in [dlLMM86], the technique did not use any representation theory. The main idea was to show that the solutions are regular along the stable/unstable leaves of an Anosov system and then, show that this implies regularity. There are a number of approaches for obtaining regularity of the solution from the regularity of the solution along transverse foliations. Besides the original one using elliptic regularity theory, we can mention Fourier series [HK90], Morrey-Campanato spaces [Jou88], Sobolev embedding [dlL04a] and Whitney regularity [dlL92, NT06]. Higher regularity for the non-commutative case was studied very thoroughly in [NT98].

In the case of cocycles taking values in a commutative group, two cocycles are cohomologous if and only if their difference is a coboundary. This does not extend to non-commutative groups and thus in this case it is natural to ask whether there is a criteria on periodic orbits to determine whether two given cocycles are cohomologous. This has been addressed in [Par99, Sch99, NT98].

We will discuss the existence of solutions in the context of cocycles taking values in Lie groups and cocycles taking values in diffeomorphism groups. Our result on cocycles taking values in a diffeomorphism group extends the earlier results of [NT95] on $\mathrm{Diff}^r(\mathbb{T}^n)$ to $\mathrm{Diff}^r(N)$ for any compact manifold N. A different approach for higher rank actions appears in [KN07].

The proof we present for the finite dimensional case is not very different from the proof of [Liv72b], but we rearrange some of the terms in the cancellations in a slightly different way so that as many of the terms are geometrically natural – only objects in the same fiber of the tangent bundle are compared. We make sure that the only comparisons which are not geometrically natural happen only in points which are

very close. Our presentation clearly illustrates the rôle played by localization assumptions. These localizations assumptions depend on the nature of group. They are always implied by η taking values in a small enough neighborhood of the identity, but they are also automatic if the group is commutative, compact or nilpotent. The behavior of non-commutative cocycle equations in the absence of such localization assumptions depends on the global geometry of the group and remains an open problem. In particular one of Livšic's original theorems [Liv72a, Theorem 3] is not justified. Resolving whether localization is necessary was posed as an open problem by Katok during the Clay Mathematics Institute and MSRI Conference on Recent Progress in Dynamics, 2004.

The rearrangement of the terms so that they are geometrically natural is not crucial in the case of Lie groups – there are many other alternative rearrangements which work – but it becomes important in the case that the group G is a diffeomorphism group. In the pioneering work [NT95], the authors needed to assume that the manifolds were essentially flat. In Section 5, we remove this assumption. The paper [NT95] also contains applications of the results on cohomology equations to rigidity of partially hyperbolic actions. If one inserts the improvements presented here on the arguments of the the argument in [NT95, NT01] one can also extend the results of those papers.

2. Some preliminaries on Anosov systems

2.1. **Definitions.** Our cocycles will be over Anosov diffeomorphisms and Anosov flows. These exhibit the strongest form of hyperbolicity, namely uniform hyperbolicity on the entire manifold.

Definition 2.1 (Anosov Diffeomorphism). *Let M be a compact Riemannian manifold. A diffeomorphism $f \in \mathrm{Diff}^r(M)$ for $r \geq 1$ is called an Anosov diffeomorphism if there exist $C > 0$ and $\lambda < 1$ and a splitting of the tangent bundle*

$$TM = E^s \oplus E^u$$

such that

(i) *For all $v \in E^s_x$ and for all $n > 0$*
$$\|Df^n_x v\| < C\lambda^n \|v\|.$$
(ii) *For all $v \in E^u_x$ and for all $n < 0$*
$$\|Df^n_x v\| < C\lambda^{|n|} \|v\|.$$

If f is an Anosov diffeomorphism with constants $C > 0$ and $\lambda < 1$ then we will call f λ-hyperbolic.

Remark 2.2. *Note that in the definition of Anosov diffeomorphism the metric enters explicitly. For a compact manifold M, if a diffeomorphism is Anosov in one metric, then it is Anosov in all metrics, and one can even take the same λ for all the metrics. The constant C, however, depends both on the metric and on the λ that we choose. If f is λ-hyperbolic then it is possible to choose a metric, as smooth as M, such that f is λ'-hyperbolic with constant $C = 1$ for any λ' with $\lambda < \lambda' < 1$. Furthermore the metric may be chosen such that the sub-bundles E^s and E^u are orthogonal. Such a metric is sometimes called an "adapted metric"* [Mat68].

Definition 2.3 (Anosov Flow)**.** *Let M be a compact Riemannian manifold. A flow $f^t : M \to M$ is called an Anosov flow if there exist $C > 0$ and $\lambda > 0$ and a splitting of the tangent bundle*

$$TM = E^s \oplus E^0 \oplus E^u$$

such that

(i) *At each $x \in M$ the subspace E^0_x is one dimensional and*
$$\frac{d}{dt} f^t(x)\Big|_{t=0} \in E^0_x \setminus \{0\}.$$
(ii) *For all $v \in E^s_x$ and for all $t > 0$*
$$\|Df^t_x v\| < Ce^{-\lambda t}\|v\|.$$
(iii) *For all $v \in E^u_x$ and for all $t < 0$*
$$\|Df^t_x v\| < Ce^{-\lambda |t|}\|v\|.$$

If f^t is an Anosov flow with constants $C > 0$ and $\lambda > 0$ then we will call f^t λ-hyperbolic.

2.2. **Anosov Foliations.** The sub-bundles $E^s, E^u \subset TM$ from the definition of Anosov diffeomorphisms and flows are called the stable and unstable bundles respectively. There are foliations W^s and W^u associated to E^s and E^u such that $T_xW^s(x) = E^s_x$ and $T_xW^u(x) = E^u_x$. These foliations can be characterized by:

$$\begin{aligned} W^s(x) &= \{y \in M : d_M(f^n(x), f^n(y)) \to 0\} \\ &= \{y \in M : d_M(f^n(x), f^n(y)) \le C_{x,y}\lambda^n,\ n > 0\} \\ W^u(x) &= \{y \in M : d_M(f^n(x), f^n(y)) \to 0 \\ &= \{y \in M : d_M(f^n(x), f^n(y)) \le C_{x,y}\lambda^{|n|},\ n < 0\} \end{aligned} \tag{2}$$

Similarly for flows one may define the center stable and center unstable bundles E^{cs} and E^{cu} by $E^{cs}_x = E^0_x \oplus E^s_x$ and $E^{cu}_x = E^0_x \oplus E^u_x$.

These are again integrable and have associated foliations W^{cs} and W^{cu} respectively.

The global structure of the stable and unstable manifolds may be quite bad – they are only immersed sub-manifolds. Moreover, though the leaves of the foliation are as smooth as the map or flow the holonomy between leaves is generally less regular than the map (the regularity is limited by ratios of contraction exponents). There are many excellent sources for the theory of invariant manifolds – see for example [HPS77] for an exposition of the Hadamard approach and see [BP06] for an exposition of the Perron approach. The original method of Poincaré was reexamined in modern language and extended in [CFdlL03]. A more comprehensive survey is [Pes04].

2.3. **The Anosov Closing Lemma.** For us the most crucial property of Anosov systems is the following shadowing lemma, often called the Anosov closing lemma.

Lemma 2.4 (Anosov Closing Lemma for Flows)**.** *Let f^t be an Anosov flow on a compact Riemannian manifold M. There exist $\epsilon_0 > 0$, $K > 0$, and $T_0 > 0$ such that if for some $T > T_0$*

$$d_M\big(f^T x, x\big) < \epsilon_0$$

we can find a unique periodic point p of period $T + \Delta$ satisfying

a) $$d_M(x,p) \le K\,\epsilon_0$$
$$d_M\big(f^T(x),p\big) \le K\,\epsilon_0$$
$$|\Delta| \le K\,\epsilon_0$$

b) $$W^s_{\mathrm{loc}}(x) \cap W^u_{\mathrm{loc}}(p) \neq \varnothing$$

Moreover, this unique point satisfies:

c) $$d_M(x,p) \le K\,d_M\big(f^T x, x\big)$$
$$d_M\big(f^T(x),p\big) \le K\,d_M\big(f^T x, x\big)$$
$$|\Delta| \le K\,d_M\big(f^T x, x\big)$$

d) $$W^s_{\mathrm{loc}}(x) \cap W^u_{\mathrm{loc}}(p) = \{z\}$$

Remark 2.5. *The statement of Lemma 2.4 is more involved than the corresponding one for diffeomorphisms, Lemma 2.6, because all the points in a periodic orbit are periodic, so that, in the case of flows, the set of periodic points of a given period, that lie in a neighborhood, is not discrete. We can hope for uniqueness of the periodic point p only if some additional condition such as b) is imposed. This is not needed in the case of diffeomorphisms, since periodic points of a fixed period are isolated. Similarly, in the case of diffeomorphisms, since the set of*

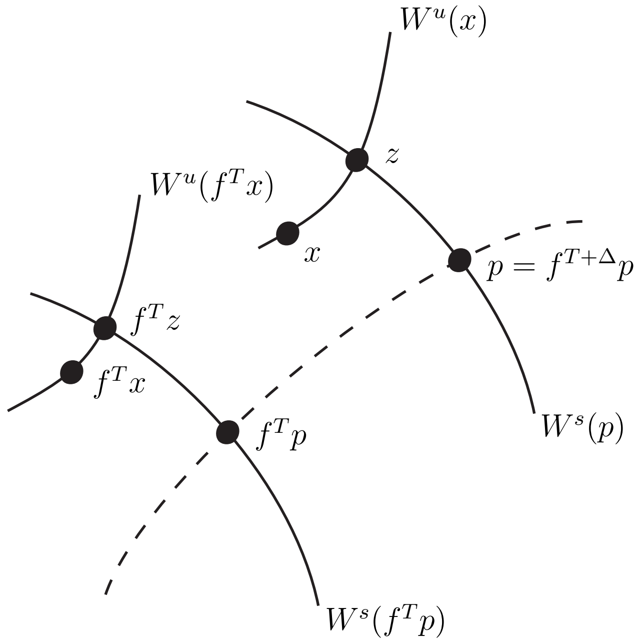

FIGURE 1. Illustration of the closing lemma, Lemma 2.4

periods is discrete, we do not have to consider the Δ that changes the period.

Lemma 2.6 (Anosov Closing Lemma for Diffeomorphisms)**.** *Let f be an Anosov diffeomorphism on a compact Riemannian manifold M. There exist $\epsilon_0 > 0$, $K > 0$, and $\lambda > 0$ such that if for some $n \in \mathbb{Z}$*

$$d_M\big(f^n x, x\big) < \epsilon_0$$

we can find a unique periodic point p of period n satisfying

a) $$d_M(x,p) \le K\,\epsilon_0$$
$$d_M\big(f^n x, p\big) \le K\,\epsilon_0$$

Moreover, this unique point satisfies:

b) $$d_M(x,p) \le K\,d_M\big(f^n x, x\big)$$
$$d_M\big(f^n x, p\big) \le K\,d_M\big(f^n x, x\big)$$

c) $$W^s_{\mathrm{loc}}(x) \cap W^u_{\mathrm{loc}}(p) = \{z\}$$

2.4. **Cocycles.**

Definition 2.7. *Let G be a group. A G-valued cocycle over a homeomorphism $f : M \to M$ is a map $\Phi : M \times \mathbb{Z} \to G$ that satisfies*

$$\Phi(x, m+n) = \Phi(f^n x, m) \cdot \Phi(x, n) \tag{3}$$

for all $x \in M$, and $m, n \in \mathbb{Z}$. Here $\cdot$ denotes the group operation.

Definition 2.8. *Let G be a group. A G-valued cocycle over a flow $f^t : M \to M$ is a map $\Phi : M \times \mathbb{R} \to G$ that satisfies*

$$\Phi(x, s+t) = \Phi(f^t x, s) \cdot \Phi(x, t). \tag{4}$$

where $x \in M$, and $s, t \in \mathbb{R}$. Here $\cdot$ denotes the group operation.

Remark 2.9. *These are special cases of the more general definition of a cocycle over a group action.*

Any cocycle Φ over a homeomorphism f is determined entirely by its generator $\eta : M \to G$ given by $\eta(x) = \Phi(x, 1)$. The cocycle Φ is reconstructed by

$$\Phi(x, n) = \begin{cases} \eta(f^{n-1}x) \cdots \eta(x) & n \geq 1 \\ \mathrm{Id} & n = 0 \\ \eta^{-1}(f^n x) \cdots \eta^{-1}(f^{-1}x) & n \leq -1 \end{cases}.$$

In the flow case the duality is not as complete. However if G is a Lie group with Lie algebra $\mathfrak{g}$, f^t is a smooth flow on M, and Φ is smooth then Φ is determined by its infinitesimal generator $\eta : M \to \mathfrak{g}$ given by

$$\eta(x) = \frac{d}{dt}\Phi(x, t)\Big|_{t=0}.$$

The cocycle can the be reconstructed as the unique solution to

$$\frac{d}{dt}\Phi(x, t) = DR_{\Phi(x,t)}\eta(f^t x), \qquad \Phi(x, 0) = \mathrm{Id}$$

where $R_{\Phi(x,t)} : g \mapsto g \cdot \Phi(x, t)$ is the operation of right multiplication by $\Phi(x, t)$, and hence $DR_{\Phi(x,t)} : \mathfrak{g} \to T_{\Phi(x,t)}G$.

3. Livšic Theory for Lie Group Valued Cocycles

Let G be a Lie group endowed with a Riemannian metric. Let d_G be the length metric on the path-connected component of the identity in G. If G is non-compact the group operation need not be Lipshitz but it is Lipshitz on any compact path-connected domain. As G is a Lie group the multiplication operator is smooth. Hence for every $g \in G$ the operators $L_h : g \mapsto h \cdot g$ and $R_h : g \mapsto g \cdot h$ are smooth.

For $F \in C^1(G, G)$ define

$$|F|_r := \max\{\|D_g F\| : g \in \overline{B}(\mathrm{Id}, r)\}$$

This gives us the following estimates

$$\text{if } d_G\big(gh^{-1}, \mathrm{Id}\big) < r \text{ then } d_G\big(g, h\big) \leq |R_h|_r \, d_G\big(gh^{-1}, \mathrm{Id}\big). \tag{5}$$

In general, without some localization, we cannot relate $d_G\big(gh^{-1}, \mathrm{Id}\big)$ and $d_G\big(g, h\big)$ but in the case of a continuous function $\eta : M \to G$ on a compact manifold M there exists $K > 0$ such that

$$\begin{aligned} d_G\big(\eta(x)\eta^{-1}(y), \mathrm{Id}\big) &< K\, d_G\big(\eta(x), \eta(y)\big) \\ d_G\big(\eta^{-1}(x)\eta(y), \mathrm{Id}\big) &< K\, d_G\big(\eta(x), \eta(y)\big) \end{aligned} \tag{6}$$

since $\eta(M)$ is a compact subset of G.

Theorem 3.1. *Let M be a compact Riemannian manifold, $f : M \to M$ be a C^1 topologically transitive λ-hyperbolic Anosov diffeomorphism, and G be a Lie group. Let $\Phi \in C^\alpha(M \times \mathbb{Z}, G)$ be a cocycle. For $x, y \in M$ define $\Delta^n_{x,y} : G \to G$ by*

$$\Delta^n_{x,y}(g) = \Phi^{-1}(x, n)\, g\, \Phi(y, n).$$

Suppose there exists $\rho > 1$ such that for all $x, y \in M$ and all $n \in \mathbb{Z}$

$$|\Delta^n_{x,y}|_{\rho^{-|n|}} \leq K\rho^{|n|}. \tag{7}$$

Suppose that that for the the pair (f, Φ):

(i) *The periodic orbit obstruction vanishes:*
If $f^n p = p$ then $\Phi(p, n) = \mathrm{Id}$.

(ii) *The hyperbolicity condition is satisfied:*

$$\rho\lambda^\alpha < 1 \tag{8}$$

then there exists $\phi \in C^\alpha(M, G)$ that solves

$$\Phi(x, n) = \phi(f^n x)\phi^{-1}(x). \tag{9}$$

Moreover, if $\hat{\phi}$ is any other continuous solution to (9) *then*

$$\hat{\phi} = \phi \cdot g$$

for some $g \in G$

Remark 3.2. *Condition 7 will be examined in greater detail in Section 4. If the group is a commutative Lie matrix group endowed with a matrix norm, or a commutative Lie group endowed with an invariant metric then no localization condition is required.*

Proof. Rearranging (9) we obtain

$$\phi(f^n x) = \Phi(x, n) \cdot \phi(x) \tag{10}$$

Thus we see that fixing $\phi(x)$ immediately determines ϕ on the entire orbit of x. Since f is topologically transitive there exists a point x^* with a dense orbit, $\mathcal{O}(x^*)$. Fixing $\phi(x^*)$ therefore defines a function $\phi : \mathcal{O}(x^*) \to G$. This shows that any continuous solution $\hat{\phi}$ to (9) is

uniquely determined by $\hat{\phi}(x^*)$. Thus choosing $g = \phi^{-1}(x^*) \cdot \hat{\phi}(x^*)$ we get

$$\hat{\phi}(x) = \phi(x) \cdot g.$$

It remains to show that the $\phi : \mathcal{O}(x^*) \to G$ defined by (10) can be extended to a C^α function $\phi : M \to G$. Standard arguments show that we can extend $\phi : \mathcal{O}(x^*) \to G$ provided it is uniformly C^α on $\mathcal{O}(x^*)$, i.e. there exists a $\delta > 0$ and $K > 0$ such that

$$\text{if } d_M(f^{n+N}x^*, f^n x^*) < \delta \text{ then} \quad d_G\big(\phi(f^{n+N}x^*), \phi(f^n x^*)\big) < K d_M(f^{n+N}x^*, f^n x^*)^\alpha. \tag{11}$$

We have the following basic estimate from (5)

$$\text{if } d_G\big(\Phi(f^n x^*, N), \mathrm{Id}\big) < 1 \text{ then} \quad d_G\big(\phi(f^{n+N}x^*), \phi(f^n x^*)\big) \le |R_{\phi(f^n x^*)}|_1 \, d_G\big(\Phi(f^n x^*, N), \mathrm{Id}\big). \tag{12}$$

First we show that the following Hölder condition on Φ,

$$\text{if } d_M(f^{n+N}x^*, f^n x^*) < \delta \text{ then} \quad d_G\big(\Phi(f^n x^*, N), \mathrm{Id}\big) < K d_M(f^{n+N}x^*, f^n x^*)^\alpha \tag{13}$$

is equivalent to the Hölder condition (11).

Choose $0 < \delta' \le \delta$ so that $K(\delta')^\alpha < 1$. The collection $\{B(f^n x^*, \delta')\}_{n\in\mathbb{Z}}$ is an open cover of M and therefore by compactness we have a finite sub-cover $\{B(f^{n_i}x^*, \delta')\}_{i=1}^m$. Let

$$L = \max_{i=1,\dots,m} |R_{\phi(f^{n_i}x^*)}|_1$$

and

$$O = \max_{i=1,\dots,m} d_G\big(\phi(f^{n_i}x^*), \mathrm{Id}\big).$$

Given an arbitrary $n \in \mathbb{Z}$ we choose $1 \le i \le m$ such that $d_M(f^n x^*, f^{n_i}x^*) < \delta'$. From (13) we get $d_G\big(\Phi(f^{n_i}x^*, n - n_i), \mathrm{Id}\big) < 1$ and hence can use (5)

$$\begin{aligned} d_G\big(\phi(f^n x^*), \mathrm{Id}\big) &\le d_G\big(\phi(f^n x^*), \phi(f^{n_i}x^*)\big) + d_G\big(\phi(f^{n_i}x^*), \mathrm{Id}\big) \\ &\le |R_{\phi(f^{n_i}x^*)}|_1 \, d_G\big(\Phi(f^{n_i}x^*, n - n_i), \mathrm{Id}\big) + d_G\big(\phi(f^{n_i}x^*), \mathrm{Id}\big) \\ &\le O + L. \end{aligned}$$

Now that we know that $\phi\big(\mathcal{O}(x^*)\big)$ is a precompact subset of G we have

$$L = \sup_{n\in\mathbb{Z}} |R_{\phi(f^n x^*)}|_1 < \infty.$$

and hence from (12)

$$\text{if } d_G\big(\Phi(f^n x^*, N), \mathrm{Id}\big) < 1 \text{ then} \\ d_G\big(\phi(f^{n+N}x^*), \phi(f^n x^*)\big) \le L\, d_G\big(\Phi(f^n x^*, N), \mathrm{Id}\big). \tag{14}$$

Then applying (13) we obtain (11). Thus to prove the theorem it suffices to prove (13).

Applying the Anosov Closing Lemma, Lemma 2.6, we obtain p with $f^N p = p$ and $d_M(f^n x^*, p) \le K d_M(f^{n+N}x^*, f^n x^*)$ and $z \in W^s(p) \cap W^u(f^n x^*)$.

We will compare the cocycle Φ along two trajectories that converge exponentially in forward time. Let $C_F(m) := \Phi^{-1}(p,m)\cdot\Phi(z,m)$. We have for $m \in \mathbb{N}$.

$$\begin{aligned} C_F(m+1) &= \Phi^{-1}(p,m)\eta^{-1}(f^m p)\cdot\eta(f^m z)\Phi(z,m) \\ &= \Delta^m_{p,z}\big(\eta^{-1}(f^m p)\cdot\eta(f^m z)\big) \end{aligned}$$

We have for $d_M(f^{n+N}x^*, f^n x^*)$ sufficiently small

$$\begin{aligned} d_G\big(\eta^{-1}(f^m p)\cdot\eta(f^m z), \mathrm{Id}\big) &\le C_1\, d_G\big(\eta(f^m p), \eta(f^m z)\big) && (6) \\ &\le C_2\, d_M(f^m p, f^m z)^\alpha && \eta \in C^\alpha(M,G) \\ &\le C_3\, \lambda^{\alpha m}\, d_M(p,z)^\alpha && z \in W^s(p) \\ &\le C_4\, \lambda^{\alpha m}\, d_M(f^{n+N}x^*, f^n x^*)^\alpha \\ &\le C_4\, \lambda^{\alpha m}\, \delta^\alpha. \end{aligned}$$

so we can choose $\delta > 0$ sufficiently small that for $m \ge 0$

$$d_G\big(\eta^{-1}(f^m p)\cdot\eta(f^m z), \mathrm{Id}\big) < \lambda^{\alpha m} < \rho^{-m}$$

and hence we can apply (7) to obtain

$$\begin{aligned} d_G\big(C_F(m+1), C_F(m)\big) &\le d_G\big(\Delta^m_{p,z}\big(\eta^{-1}(f^m p)\cdot\eta(f^m z)\big), \Delta^m_{p,z}\big(\mathrm{Id}\big)\big) \\ &\le |\Delta^m_{p,z}|_{\rho^{-m}}\, d_G\big(\eta^{-1}(f^m p)\cdot\eta(f^m z), \mathrm{Id}\big) \end{aligned}$$

Thus

$$\begin{aligned} d_G\big(C_F(m+1), C_F(m)\big) &\le |\Delta^m_{p,z}|_{\rho^{-m}}\, C_4\, \lambda^{\alpha m}\, d_M(f^{n+N}, f^n x^*)^\alpha \\ &\le C_5\, (\rho\lambda^\alpha)^m\, d_M(f^{n+N}x^*, f^n x^*)^\alpha \end{aligned}$$

By using the hyperbolicity assumption (8) we get the following bound

$$d_G\big(C_F(m), \mathrm{Id}\big) \le \frac{C_5}{1-\rho\lambda^\alpha}\, d_M(f^{n+N}x^*, f^n x^*)^\alpha.$$

In particular, since $\Phi(p,N)=\mathrm{Id}$, we have $C_F(N)=\Phi(z,N)$ and hence obtain

$$d_G\big(\Phi(z,N),\mathrm{Id}\big)\le \frac{C_5}{1-\rho\lambda^\alpha}\,d_M(f^{n+N}x^*,f^nx^*)^\alpha. \tag{15}$$

This means that $d_G\big(\Phi(z,N),\mathrm{Id}\big)$ is uniformly bounded and consequently $|R_{\Phi(z,N)}|_1$ is uniformly bounded.

Now we compare Φ along two trajectories that converge exponentially in backwards time. For $m\in\mathbb{N}$ we define

$$\begin{aligned}C_R(m)&=\Phi^{-1}(f^{n+N}x^*,-m)\cdot\Phi(f^Nz,-m)\\&=\Delta^{-m}_{f^{n+N}x^*,f^Nz}(\mathrm{Id})\end{aligned}$$

From the definition of the cocycle we obtain

$$C_R(m+1)=\Delta^{-m}_{f^{n+N}x^*,f^Nz}\big(\eta(f^{n+N-m-1}x^*)\eta^{-1}(f^{N-m-1}z)\big).$$

Exactly as before, we are able to estimate

$$d_G\big(C_R(m+1),C_R(m)\big)\le C_5\,(\rho\lambda^\alpha)^m\,d_M(f^{n+N}x^*,f^nx^*)^\alpha.$$

By the hyperbolicity assumption we have $\rho\lambda^\alpha<1$ and hence,as before, we get the following bound, uniform in m

$$d_G\big(C_R(m),\mathrm{Id}\big)\le\frac{C_5}{1-\rho\lambda^\alpha}d_M(f^{n+N}x^*,f^nx^*)^\alpha.$$

In particular, for $m=N$

$$d_G\big(C_R(N),\mathrm{Id}\big)\le\frac{C_5}{1-\rho\lambda^\alpha}d_M(f^{n+N}x^*,f^nx^*)^\alpha. \tag{16}$$

From the cocycle property we obtain

$$\Phi(f^Nz,-N)=\Phi^{-1}(z,N)$$

and

$$\Phi^{-1}(f^{N+n}x^*,-N)=\Phi(f^nx^*,N).$$

Thus

$$\begin{aligned}&d_G\big(\Phi(f^nx^*,N),\mathrm{Id}\big)\\&\qquad\le d_G\big(\Phi^{-1}(f^{n+N}x^*,-N)\Phi(f^Nz,-N)\Phi(z,N),\Phi(z,N)\big)\\&\qquad\qquad+d_G(\Phi(z,N),\mathrm{Id})\\&\qquad\le|R_{\Phi(z,N)}|_1\,d_G\big(C_R(N),\mathrm{Id}\big)+d_G\big(\Phi(z,N),\mathrm{Id}\big).\end{aligned}$$

From (15), the fact $|R_{\Phi(z,N)}|_1$ is uniformly bounded, and (16) we get

$$d_G\big(\Phi(f^nx^*,N),\mathrm{Id}\big)\le L\,d_M(f^{n+N}x^*,f^nx^*)^\alpha$$

which establishes (13) and hence completes the proof. $\square$

Remark 3.3. *The hyperbolicity condition* (7) *we require is stronger than what we actually use. We could make do with the more complicated condition: there exists $\rho > 1$ such that*

(i) *for all $x \in M$, all $y \in W^s(x)$, and all $n \geq 0$*

$$|\Delta^n_{x,y}|_{\rho^{-n}} \leq K\rho^n.$$

(ii) *for all $x \in M$, all $y \in W^u(x)$, and all $n \leq 0$*

$$|\Delta^n_{x,y}|_{\rho^n} \leq K\rho^{-n}.$$

This more complicated hyperbolicity condition is useful in the case of commutative groups endowed with matrix norms.

We now give a similar proof for Lie group valued cocycles over Anosov flows.

Theorem 3.4. *Let M be a compact Riemannian manifold, $f^t : M \to M$ be a C^1 topologically transitive λ-hyperbolic Anosov flow, and G be a Lie group. Let $\eta \in C^\alpha(M, \mathfrak{g})$.*

Define the cocycle $\Phi : M \times \mathbb{R} \to G$ by

$$\frac{d}{dt}\Phi(x,t) = DR_{\Phi(x,t)}\eta(f^t x), \qquad \Phi(x,0) = \mathrm{Id}$$

and let $\Delta^t_{x,y} : \mathfrak{g} \to T_{\Phi^{-1}(x,t)\Phi(y,t)}G$ be given by

$$\Delta^t_{x,y} = DL_{\Phi^{-1}(x,t)} DR_{\Phi(y,t)}.$$

Assume the following localization condition; there exist $K, \rho > 0$ such that for all $x.y \in M$

$$\|\Delta^t_{x,y}\| \leq K e^{\rho|t|} \tag{17}$$

where $\|\cdot\|$ is the standard operator norm. Suppose that for the pair f^t and η:

(i) *The periodic orbit obstruction vanishes:*
If $f^t p = p$ then $\Phi(p,t) = \mathrm{Id}$.

(ii) *The hyperbolicity condition is satisfied:*

$$\rho - \lambda\alpha < 0. \tag{18}$$

Then there exists $\phi \in C^\alpha(M, G)$ that solves

$$\Phi(x,t) = \phi(f^t x)\phi^{-1}(x). \tag{19}$$

Proof. Let $x^* \in M$ be a point with a dense orbit $\mathcal{O}(x^*)$. If we fix $\phi(x^*)$ then, by (19), we can define ϕ on $\mathcal{O}(x^*)$ by

$$\phi(f^t x^*) = \Phi(x^*, t)\,\phi(x^*).$$

Exactly as in the previous case, it suffices to show that there exist $\delta > 0$ and $K > 0$ such that

$$\text{if } d_M(f^{t+T}x^*, f^t x^*) < \delta \text{ then} \quad d_G\big(\Phi(f^t x^*, T), \mathrm{Id}\big) < K d_M(f^{t+T}x^*, f^t x^*)^\alpha. \tag{20}$$

Applying the Anosov Closing Lemma, Lemma 2.4, with $d_M(f^{t+T}x^*, f^t x^*) < \delta$ we obtain a periodic point $p \in M$ with $f^{T+\Delta}p = p$ and a point $z \in W^s(p) \cap W^u(f^t x^*)$. The periodic point satisfies:

(i) $|\Delta| < K\, d_M(f^{t+T}x^*, f^t x^*)$.
(ii) $d_M(f^{t+T}x^*, p) \le K\, d_M(f^{t+T}x^*, f^t x^*)$.

Let $C_F(s) := \Phi^{-1}(p,s) \cdot \Phi(z,s)$. Using the chain rule for functions of two variables we obtain

$$\begin{aligned} \frac{d}{ds} C_F(s) &= DL_{\Phi^{-1}(p,s)} DR_{\Phi(z,s)} \big[\eta(f^s z) - \eta(f^s p)\big] \\ C_F(0) &= 0 \end{aligned} \tag{21}$$

From the definition of $\Delta_{p,z}$ we obtain

$$\frac{d}{ds} C_F(s) = \Delta^s_{p,z} \big[\eta(f^s z) - \eta(f^s p)\big]$$

and hence, for $s > 0$, we have the estimate

$$\begin{aligned} \left\| \frac{d}{ds} C_F(s) \right\| &\le \|\Delta^s_{p,z}\|\, \|\eta(f^s z) - \eta(f^s p)\| \\ &\le C_1\, e^{\rho s} \|\, \eta(f^s z) - \eta(f^s p)\| && (17) \\ &\le C_2\, e^{\rho s}\, d_M(f^s z, f^s p)^\alpha && \eta \in C^\alpha(M, \mathfrak{g}) \\ &\le C_3\, e^{(\rho - \lambda\alpha)s}\, d_M(z,p)^\alpha && z \in W^s(p) \\ &\le C_4\, e^{(\rho - \lambda\alpha)s}\, d_M(f^{t+T}x^*, f^t x^*)^\alpha && \text{Lemma 2.4} \end{aligned}$$

As $C_F(0) = \mathrm{Id}$, using the hyperbolicity assumption (18), and the fact d_G is a length metric we can integrate to get the following bound

$$d_G\big(C_F(s), \mathrm{Id}\big) \le \frac{C_4}{\lambda\alpha - \rho} d_M(f^{t+T}x^*, f^t x^*)^\alpha \tag{22}$$

and in particular

$$\begin{aligned} d_G\big(C_F(T), \mathrm{Id}\big) &= d_G\big(\Phi^{-1}(p,T)\Phi(z,T), \mathrm{Id}\big) \\ &\le \frac{C_4}{\lambda\alpha - \rho} d_M(f^{t+T}x^*, f^t x^*)^\alpha \end{aligned}$$

As the periodic orbit obstruction is satisfied we have $\Phi(p, T+\Delta) = \mathrm{Id}$. By the cocycle property

$$\Phi(p,T) = \Phi(p, -\Delta)$$

From the Anosov Closing Lemma, Lemma 2.4, we have $|\Delta| < K\, d_M(f^{t+T}x^*, f^t x^*)$ and hence Δ is bounded. By compactness we can uniformly bound

$$|L_{\Phi(p,T)}|_1 = |L_{\Phi(p,-\Delta)}|_1 < C$$

Now we estimate

$$\begin{aligned} d_G\big(\Phi(z,T),\mathrm{Id}\big) &\le d_G\big(\Phi(p,T)\cdot\Phi^{-1}(p,T)\cdot\Phi(z,T),\Phi(p,T)\big) \\ &\qquad + d_G\big(\Phi(p,T),\mathrm{Id}\big) \\ &\le |L_{\Phi(p,T)}|_1\, d_G\big(C_F(T),\mathrm{Id}\big) + d_G\big(\Phi(p,T),\mathrm{Id}\big) \\ &\le C\, d_M(f^{t+T}x^*, f^t x^*)^\alpha. \end{aligned} \tag{23}$$

Since $d_G\big(\Phi(z,T),\mathrm{Id}\big)$ is bounded we can find a uniform estimate for $|R_{\Phi(z,T)}|_1$.

Let $C_R(s) := \Phi^{-1}(f^{t+T}x^*,-s)\Phi(f^T z,-s)$. We have

$$\begin{aligned} \frac{d}{ds}C_R(s) &= DL_{\Phi^{-1}(f^{t+T}x^*,-s)}DR_{\Phi(f^T z,-s)}\big[\eta(f^{t+T-s}x^*)-\eta(f^{T-s}z)\big] \\ &= \Delta^{-s}_{f^{t+T}x^*,f^T z}\big[\eta(f^{t+T-s}x^*)-\eta(f^{T-s}z)\big] \end{aligned}$$

For $s>0$, we have

$$\begin{aligned} \Big\|\frac{d}{ds}C_R(s)\Big\| &\le \|\Delta^{-s}_{f^{t+T}x^*,f^T z}\|\,\|\eta(f^{t+T-s}x^*)-\eta(f^{T-s}z)\| \\ &\le C_1\, e^{\rho s}\|\,\eta(f^{t+T-s}x^*)-\eta(f^{T-s}z)\| && (17) \\ &\le C_2\, e^{\rho s}\, d_M(f^{t+T-s}x^*, f^{T-s}z)^\alpha && \eta\in C^\alpha(M,\mathfrak{g}) \\ &\le C_3\, e^{(\rho-\lambda\alpha)s}\, d_M(f^{t+T}x^*, f^T z)^\alpha && f^T z\in W^u(f^{t+T}x^*) \\ &\le C_4\, e^{(\rho-\lambda\alpha)s}\, d_M(f^{t+T}x^*, f^t x^*)^\alpha && \text{Lemma 2.4.} \end{aligned}$$

By the hyperbolicity assumption we have $\rho-\lambda\alpha<0$ and hence we get the following bound, uniform in s

$$d_G\big(C_R(s),\mathrm{Id}\big) \le \frac{C_4}{\lambda\alpha-\rho} d_G(f^{t+T}x^*, f^t x^*)^\alpha.$$

In particular, for $s=T$

$$d_G\big(C_R(T),\mathrm{Id}\big) \le \frac{C_4}{\lambda\alpha-\rho} d_G(f^{t+T}x^*, f^t x^*)^\alpha. \tag{24}$$

Using the cocycle property we can rewrite $C_R(T)$ in the form

$$\begin{aligned} C_R(T) &= \Phi^{-1}(f^{T+t}x^*,-T)\Phi(f^T z,-T) \\ &= \Phi(f^t x^*,T)\Phi^{-1}(z,T) \end{aligned}$$

Finally using the triangle inequality, (23), and (24) we get

$$\begin{aligned} d_G\big(\Phi(f^t x^*, T), \mathrm{Id}\big) \le\ & d_G\big(\Phi(f^t x^*, T)\Phi^{-1}(z, T)\Phi(z, T), \Phi(z, T)\big) \\ & \quad + d_G\big(\Phi(z, T), \mathrm{Id}\big) \\ \le\ & |R_{\Phi(z,T)}|_1\, d_G\big(C_R(T), \mathrm{Id}\big) + d_G\big(\Phi(z, T), \mathrm{Id}\big) \\ \le\ & C\, d_M(f^{t+T}x^*, f^t x^*)^\alpha. \end{aligned}$$

This completes the proof of Theorem 3.4. □

One may deduce a version of the result for Anosov diffeomorphisms, Theorem 3.1, from the one for Anosov flows, Theorem 3.4, by a suspension trick.

Remark 3.5. *The hyperbolicity condition* (17) *we require is stronger than what we actually use. We could make do with the more complicated condition: there exists $\rho > 1$ such that*

(i) *for all $x \in M$, all $y \in W^s(x)$, and all $t \ge 0$*

$$\|\Delta^t_{x,y}\| \le K e^{\rho t}$$

(ii) *for all $x \in M$, all $y \in W^u(x)$, and all $t \le 0$*

$$\|\Delta^t_{x,y}\| \le K e^{-\rho t}$$

This more complicated hyperbolicity condition is useful in the case of commutative groups endowed with matrix norms.

4. Verifying Localization

The localization conditions (7) and (17) are formulated without any assumptions on the metric. This has the advantage that the arguments apply equally well to matrix norms, useful in computations in matrix Lie groups, and to the left-invariant (or right-invariant) metrics so useful in geometric computations. Finally these arguments also shed light on cases such as diffeomorphism groups where the natural metric lacks the special properties of either matrix norms or invariant metrics.

In the case of matrix norms and invariant metrics we can easily relate the localization conditions (7) and (17) to properties of the generating map or vector field.

4.1. **Localization in Matrix Norms.**

4.1.1. *Diffeomorphism Case.* Let G be a matrix Lie group endowed with a matrix norm. We will use only the multiplicative property

$$\|AB\| \leq \|A\| \, \|B\|.$$

The operator $\Delta^n_{x,y}(g)$ satisfies

$$\begin{aligned} \|\Delta^n_{x,y}(g)\| &= \|\Phi^{-1}(x,n)\, g\, \Phi(y,n)\| \\ &\leq \|\Phi^{-1}(x,n)\| \, \|\Phi(y,n)\| \, \|g\| \end{aligned}$$

and hence we have

$$|\Delta^n_{x,y}| \leq \|\Phi^{-1}(x,n)\| \, \|\Phi(y,n)\|.$$

If we let $\rho^2 = \max_{x\in M}\{\|\eta(x)\|, \|\eta^{-1}(x)\|\}$ then we have

$$\|\Phi(x,n)\| \leq \rho^{\frac{1}{2}|n|} \qquad \|\Phi^{-1}(x,n)\| \leq \rho^{\frac{1}{2}|n|}$$

and hence

$$|\Delta^n_{x,y}(g)| \leq \rho^{|n|}.$$

4.1.2. *Flow Case.* Similarly, the operator $\Delta^t_{x,y}v$ satisfies

$$\begin{aligned} \|\Delta^t_{x,y}v\| &= \|DL_{\Phi^{-1}(x,t)} DR_{\Phi(y,t)} v\| \\ &\leq \|\Phi^{-1}(x,t)\| \, \|\Phi(y,t)\| \, \|v\| \end{aligned}$$

and hence we have

$$\|\Delta^t_{x,y}\| \leq \|\Phi^{-1}(x,t)\| \, \|\Phi(y,t)\|.$$

If we let $2\rho = \max_{x\in M}\{\|\eta(x)\|, \|\eta^{-1}(x)\|\}$ then we have

$$\|\Phi(x,t)\| \leq e^{\frac{\rho}{2}|t|} \qquad \|\Phi^{-1}(x,t)\ | \leq e^{\frac{\rho}{2}|t|}$$

and hence

$$|\Delta^t_{x,y}v| \leq e^{\rho|t|}$$

4.1.3. *Commutative Matrix Groups.* In the case of a commutative matrix group no localization assumption is required. First observe that for commutative matrix groups we have

$$|\Delta^n_{x,y}| \leq \|\Phi^{-1}(x,n)\Phi(y,n)\|, \qquad \|\Delta^t_{x,y}\| \leq \|\Phi^{-1}(x,t)\Phi(y,t)\|.$$

Thus the key is to estimate the quantity $\|\Phi^{-1}(x,n)\Phi(y,n)\|$ in the case of an Anosov diffeomorphism, or the quantity $\|\Phi^{-1}(x,t)\Phi(y,t)\|$ in the case of an Anosov flow.

In the case of a cocycle over an Anosov diffeomorphism we have the evolution equation

$$\|\Phi^{-1}(x,n+1)\Phi(y,n+1)\| \leq \|\eta^{-1}(f^n x)\eta(f^n y)\| \, \|\Phi^{-1}(x,n)\Phi(y,n)\|.$$

For $x \in M$, $y \in W^s(x)$, and $n \geq 0$ we have

$$\|\eta^{-1}(f^n x)\eta(f^n y)\| \leq 1 + D\lambda^{\alpha n}.$$

You can easily verify by induction that in this case for all $n \geq 0$ we have

$$\|\Phi^{-1}(x,n)\Phi(y,n)\| \leq e^{D\frac{1-\lambda^(\alpha\, n)}{1-\lambda^\alpha}}$$

and hence $|\Phi^{-1}(x,n)\Phi(y,n)\| < e^{\frac{D}{1-\lambda}}$. A similar computation works when $x \in M$, $y \in W^u(x)$, and $n \leq 0$.

In the case of flows we have the following evolution equation

$$\frac{d}{dt}\|\Phi^{-1}(x,t)\Phi(y,t)\| \leq \|\eta(f^t y) - \eta(f^t x)\| \, \|\Phi^{-1}(x,t)\Phi(y,t)\|.$$

For $x \in M$, $y \in W^s(x)$, and $t > 0$ we have

$$\|\eta(f^t y) - \eta(f^t x)\| < D\, e^{-\alpha\,\lambda\, t}.$$

Check that in this case a version of the Gronwall inequality gives us

$$\|\Phi^{-1}(x,t)\Phi(y,t)\| \leq e^{\frac{D}{\alpha\,\lambda} e^{-\alpha\,\lambda\, t}}$$

and hence $\|\Phi^{-1}(x,t)\Phi(y,t)\| \leq e^{\frac{D}{\alpha\,\lambda}}$. A similar computation works when $x \in M$, $y \in W^u(x)$, and $t \leq 0$.

4.2. **Localization in Right Invariant Norms.** A metric $d_G$on a topological group G is called right invariant if for all $f, g, h \in G$, $d_G(f \cdot h, g \cdot h) = d_g(f, g)$. First observe that the local Lipshitz constant of the left multiplication operator, and the operator norm of the differential map of the left multiplication operator, are independent of the base point since the metric is invariant under right multiplication and left and right multiplication commute.

4.2.1. *Diffeomorphism Case.* If we let

$$\rho = \max_{x \in M} \max\{|L_{\eta(x)}|_1, |L_{\eta^{-1}(x)}|_1\}$$

then by definition of $\Phi(x,n)$ we can write

$$\begin{aligned}
|L_{\Phi(x,n)}|_{\rho^{-|n|}} &= \begin{cases} |L_{\eta(f^{n-1}x)} \circ \cdots \circ L_{\eta(x)}|_{\rho^{-|n|}} & n > 0 \\ 1 & n = 0 \\ |L_{\eta(f^n x)} \circ \cdots \circ L_{\eta(f^{-1}x)}|_{\rho^{-|n|}} & n < 0 \end{cases} \\
&\leq \begin{cases} |L_{\eta(f^{n-1}x)}|_1 \cdots |L_{\eta(x)}|_1 & n > 0 \\ 1 & n = 0 \\ |L_{\eta^{-1}(f^n x)}|_1 \cdots |L_{\eta^{-1}(f^{-1}x)}|_1 & n < 0 \end{cases} \\
&\leq \rho^{|n|}.
\end{aligned}$$

Since R_g is an isometry the same estimate holds for our operator $\Delta^n_{x,y}$ so

$$|\Delta^n_{x,y}|_{\rho^{-|n|}} \leq \rho^{|n|}.$$

4.2.2. *Flow Case.* Observe that since the metric is right invariant

$$\|\Delta^t_{x,y}\| = \|D_e L_{\Phi^{-1}(x,t)}\|.$$

If we let

$$\rho := \max_{x\in M} \max_{t\in[-1,1]} \log \|DL_{\Phi^{-1}(x,t)}\|$$

then

$$\|\Delta^t_{x,y}\| \leq e^{\rho\lceil|t|\rceil} \leq e^{\rho} e^{\rho|t|}.$$

4.2.3. *Commutative Group.* In a commutative group a right invariant metric is simply invariant and hence the operators $\Delta^n_{x,y}$ and $\Delta^t_{x,y}$ are isometries. Hence we can take $\rho = 0$ and the hyperbolicity conditions are automatically satisfied.

5. Livšic Theory in Diffeomorphism Groups

5.1. **Preliminaries on Diffeomorphism Groups.** .

We will recall some of the standard material on global analysis, see for example [Ban97].

We consider cocycles taking values in the group of C^r diffeomorphisms of a compact Riemannian manifold N. The group operation is composition. As it is well known, the group operation is continuous but not differentiable [dlLO99]. Hence the previous results do not apply directly. Nevertheless, we will see that the rough lines of the technique can be applied, but we get some lower regularity of the solutions.

The group of C^r diffeomorphisms of a compact Riemannian manifold N has the structure of a Banach manifold modeled on the space $T_hC^r(N,N)$, defined by

$$T_hC^r(N,N) = \{v \in C^r(N,TN) : \pi_N \circ v = h\}.$$

We will endow with $\mathrm{Diff}^r(N)$ with a length metric induced from the Riemannian structure on N. Given $h \in \mathrm{Diff}^r(N)$ and $y \in N$ there exists a neighborhood $U \subset T_yN$ sufficiently small that a local representative $\tilde{h}_y : U \to T_{h(y)}N$ is uniquely defined by

$$h\big(\exp_y v\big) = \exp_{h(y)}\big(\tilde{h}_y(v)\big).$$

Since $\tilde{h}_y$ is defined between Banach spaces we can differentiate it in the usual manner. We will always use D to denote differentiation in the manifold N.

We thus obtain

$$D^n h(y) := D^n \tilde{h}(0) : (T_y N)^{\otimes n} \to T_{h(y)} N.$$

The derivative produced in this fashion coincides with the usual notion of covariant derivative defined by the Levi-Civita connection.

When dealing with a smooth curve $h : \mathbb{R} \to \mathrm{Diff}^r(N)$ we modify this idea slightly. For any $s \in \mathbb{R}$ and any $y \in N$ there exists a neighborhood V of s and $0 \in U \subset T_y N$ such that for any $t \in V$ the local representative $\tilde{h}(t)_y : U \subset T_y N \to T_{h(s)(y)} N$ is defined uniquely by

$$h(t)\big(\exp_y v\big) = \exp_{h(s)(y)}\big(\tilde{h}(t)_y(v)\big).$$

We may therefore differentiate with respect to t to obtain

$$\frac{d}{dt}\tilde{h}(t)_y\Big|_{t=s} : U \subset T_y N \to T_{h(s)(y)} N.$$

We declare

$$\frac{d}{ds} D_y^n h(s) := D^n \frac{d}{dt}\tilde{h}(t)_y\Big|_{t=s}(0).$$

5.2. **Metric on** $\mathrm{Diff}^r(N)$**.** Let $p : [0,1] \to \mathrm{Diff}^r(N)$ be a piecewise C^1 path. This is equivalent to $\frac{d}{ds} D^n p_s$ piecewise continuous in s for $0 \le n \le r$. We can define the length of such a piecewise C^1 path by

$$\ell_r(p) = \max_{0 \le n \le r} \max_{y \in N} \int_0^1 \| \frac{d}{ds} D_y^n p_s \| \, ds$$

where the norm is the appropriate operator norm induced by the Riemannian metric. If we compute the length of only a part of the path then we write

$$\ell_r(p; s) = \max_{0 \le n \le r} \max_{y \in N} \int_0^s \| \frac{d}{dt} D_y^n p_t \| \, dt$$

When we wish to compute $\frac{d}{dt}\|D_y^k p_t\|$ we can use the local representative but must take care to consider the lift of the appropriate norm. Let $p \in N$ be an arbitrary point and q close enough to p that we may consider the lift of q to the neighborhood $U \subset T_p N$ on which the local representative is defined. Since the Riemannian metric is smooth we can find a globally defined constant $\kappa > 0$, depending on the Riemannian metric, such that

$$\| \cdot \|_q \le \| \cdot \|_p + \kappa \, \| \cdot \|_p \, d_N(q, p). \tag{25}$$

In fact we will only need the infinitesimal version of this.

Notice

$$\ell_0(p) = \max_{y \in N} \int_0^1 \| \frac{d}{ds} p_s(y) \|_{p_s(y)} \, ds$$

is precisely the maximum over all $y \in N$ of the usual length of the path $p_s(y)$ in N. We use this length structure on $\mathrm{Diff}^r(N)$ to induce a metric by defining

$$d_r(g,h) := \inf_{p\in\mathfrak{P}} \max\{\ell(p), \ell(p^{-1})\} \tag{26}$$

where

$$\mathfrak{P} = \big\{p \in C^1_{\mathrm{pw}}\big([0,1], \mathrm{Diff}^r(N)\big), p_0 = g, p_1 = h\big\}.$$

Notice that our definition has the symmetry property

$$d_r(g,h) = d_r(g^{-1}, h^{-1}).$$

It is worth noticing that for paths which connect a diffeomorphism f to the identity $\ell_0(p) = \ell_0(p^{-1})$. Furthermore, if f and g are sufficiently C^1 close there is a standard way of producing an interpolating path, namely

$$p_s(y) := \exp_{f(y)}\big(s \exp^{-1}_{f(y)} g(y)\big).$$

Since geodesics are locally distance minimizing for f and g sufficiently close this path is the path along which d_0 is minimized. In this case, we have

$$d_0(f,g) = \max\big\{\max_{y\in N} d_N\big(f(y), g(y)\big), \max_{y\in N} d_N\big(f^{-1}(y), g^{-1}(y)\big)\big\}.$$

If f and g are not sufficiently close then this may no longer necessarily true.

Our first lemma gives explicit estimates on the size of derivatives in an interpolating path.

Lemma 5.1. *For all piecewise C^1 paths $p : \mathbb{R} \to \mathrm{Diff}^r(N)$, all $1 \le k \le r$, and all $s > 0$*

$$\|D^k p_s\| \le e^{\kappa\, \ell_0(p;s)} \left(\ell_k(p;s) + \|D^k p_0\|\right)$$

where the norms are the operator norms for $D^k p_s, D^k p_0 : (TN)^{\otimes k} \to TN$, and κ is the constant from (25). In particular

$$\|Dp_s\|_{r-1} \le e^{\kappa\, \ell_0(p;s)} \left(\ell_r(p;s) + \|Dp_0\|_{r-1}\right)$$

Proof. Let $0 \le k \le r$. Let $y \in N$ and $v \in (T_yN)^{\otimes k}$ be arbitrary. The estimate (25) gives us

$$\begin{aligned}\|D^k_y p_{t+\epsilon} v\|_{p_{t+\epsilon}(y)}& \\ &\le \|D^k_y p_{t+\epsilon} v\|_{p_t(y)} + \kappa \, \|D^k_y p_{t+\epsilon} v\|_{p_t(y)}\, d_N\big(p_{t+\epsilon}(y), p_t(y)\big)\end{aligned}$$

and hence

$$\begin{aligned}\|D^k_y p_{t+\epsilon} v\|_{p_{t+\epsilon}(y)} - \|D^k_y p_t v\|_{p_t(y)} &\le \|D^k_y p_{t+\epsilon} v\|_{p_t(y)} - \|D^k_y p_t v\|_{p_t(y)} \\ &\quad + \kappa\, \|D^k_y p_{t+\epsilon} v\|_{p_t(y)}\, d_N\big(p_{t+\epsilon}(y), p_t(y)\big).\end{aligned}$$

Since the terms on the right hand side have the same base point the triangle inequality applies

$$\|D^k_y p_{t+\epsilon} v\|_{p_t(y)} - \|D^k_y p_t v\|_{p_t(y)} \le \|D^k_y p_{t+\epsilon} v - D^k_y p_t v\|_{p_t(y)}$$

Dividing by ϵ and taking the limit we obtain

$$\frac{d}{dt}\|D^k_y p_t v\|_{p_t(y)} \le \|\frac{d}{dt} D^k_y p_t v\|_{p_t(y)} + \kappa\, \|D^k_y p_t v\|_{p_t(y)} \|\frac{d}{dt} p_t(y)\|_{p_t(y)}.$$

The classical Gronwall inequality therefore gives us

$$\begin{aligned}\|D^k_y p_s v\|_{p_s(y)} &\le e^{\kappa \int_0^s \|\frac{d}{dt} p_t(y)\|_{p_t(y)}\, dt} \Big(\int_0^s \|\frac{d}{dt} D^k_y p_t v\|_{p_t(y)}\, dt + \|D^k_y p_0 v\|_{p_0(y)}\Big)\\ &\le e^{\kappa\, \ell_0(p;s)} \big(\ell_k(p;s) + \|D^k_y p_0 v\|_{p_0(y)}\big)\end{aligned}$$

Finally we take a supremum over all $v \in (T_y N)^{\otimes k}$ with $\|v\|_y = 1$ and then a supremum over $y \in N$. □

Our second lemma contains the central part of a version of the mean value theorem for our metric.

Lemma 5.2. *Let $p \in C^1\big([0,1], \mathrm{Diff}^{r-1}(N)\big)$ and let $h \in \mathrm{Diff}^r(N)$. Then,*

$$\ell_{r-1}(h \circ p_s) \le C\, \|Dh\|_{r-1}\, (1 + \max_{s\in[0,1]} \|Dp_s\|_{r-2})^{r-1}\, \ell_{r-1}(p_s).$$

Let $p \in C^1\big([0,1], \mathrm{Diff}^r(N)\big)$ and let $h \in \mathrm{Diff}^r(N)$. Then,

$$\ell_r(p_s \circ h) \le C \max_{k_1,\cdots,k_r} \|D^1 h\|^{k_1} \cdots \|D^r h\|^{k_r}\, \ell_r(p_s)$$

where the max is taken over all $k_1, \ldots, k_r \ge 0$ such that

$$k_1 + 2k_2 + \cdots + r k_r \le r.$$

Crudely, this may be estimated by

$$\ell_r(p_s \circ h) \le C\, (1 + \|Dh\|_{r-1})^r\, \ell_r(p_s).$$

In each case, the constant C depends on r.

Proof. To determine $\ell_r(p_s \circ h)$ we need to compute $\frac{d}{ds} D^n[p_s \circ h]$ for $0 \le n \le r$. We apply the Faà di Bruno formula to $p_s \circ h$ to obtain

$$D^n[p_s \circ h] = \sum_{k_1,\ldots,k_n} D^k p_s \circ h \cdot [D^1 h^{\otimes k_1} \otimes \cdots \otimes D^n h^{\otimes k_n}]$$

where $k = k_1 + \cdots + k_n$ and the sum is taken over all $k_1, \ldots, k_n$ such that $k_1 + 2k_2 + \cdots + n k_n = n$. We use $\circ$ to denote composition in

the base space N and $\cdot$ to indicate composition (multiplication) in the space of linear operators. Differentiating with respect to s, we obtain

$$\frac{d}{ds}D^n[p_s \circ h] = \sum_{k_1,\dots,k_n} C_{k_1,\dots,k_n}\,\frac{d}{ds}D^k p_s \circ h \cdot \left[D^1 h^{\otimes k_1} \otimes \cdots \otimes D^n h^{\otimes k_n}\right].$$

We have the following estimate

$$\begin{aligned}\left\|D^1 h^{\otimes k_1} \otimes \cdots \otimes D^n h^{\otimes k_n}\right\| &= \|D^1 h\|_0^{k_1} \cdots \|D^n h\|_0^{k_n}\\ &\le (1 + \|Dh\|_{n-1})^n\\ &\le (1 + \|Dh\|_{r-1})^r.\end{aligned}$$

Since

$$\int_0^1 \|\frac{d}{ds}D^k p_s \circ h\|\, ds \le \ell_k(p_s) \le \ell_n(p_s) \le \ell_r(p_s)$$

we combine our estimates to obtain

$$\ell_r(p_s \circ h) \le C\,(1 + \|Dh\|_{r-1})^r \ell_r(p_s).$$

The constant C depends only on r.

To determine $\ell_{r-1}(h \circ p_s)$ we need to compute $\frac{d}{ds}D^n[h \circ p_s]$ for $0 \le n \le r-1$. We apply the Faà di Bruno formula to $h \circ p_s$ to obtain

$$D^n[h \circ p_s] = \sum_{k_1,\dots,k_n} C_{k_1,\dots,k_n}\, D^k h \circ p_s \cdot [D^1 p_s^{\otimes k_1} \otimes \cdots \otimes D^n p_s^{\otimes k_n}]$$

where $k = k_1 + \cdots + k_n$ and the sum is taken over all $k_1, \dots, k_n$ such that $k_1 + 2k_2 + \cdots + nk_n = n$. Differentiating with respect to s, we obtain

$$\begin{aligned}\frac{d}{ds}D^n[h \circ p_s] = \sum_{k_1,\dots,k_n} \big[&D^{k+1} h \circ p_s \cdot [\frac{d}{ds}p_s \otimes D^1 p_s^{\otimes k_1} \otimes \cdots \otimes D^n p_s^{\otimes k_n}] +\\ &D^k h \circ p_s \cdot \frac{d}{ds}[D^1 p_s^{\otimes k_1} \otimes \cdots \otimes D^n p_s^{\otimes k_n}]\big]\end{aligned}$$

The term

$$\frac{d}{ds}[D^1 p_s^{\otimes k_1} \otimes \cdots \otimes D^n p_s^{\otimes k_n}]$$

consists of k terms, each of which has a single term of the form $\frac{d}{ds}D^l p_s$ and thus can be estimated by

$$\|D^1 p_s\|^{k_1} \cdots \|D^l p_s\|^{k_l - 1} \cdots \|D^n p_s\|^{k_n}\,\big\|\frac{d}{ds}D^l p_s\big\|.$$

As above we can estimate

$$\begin{aligned}&\|D^1 p_s\|^{k_1} \cdots \|D^l p_s\|^{k_l - 1} \cdots \|D^n p_s\|^{k_n}\\ &\qquad\le (1 + \max_{s\in[0,1]} \|Dp_s\|_{n-1})^n \le (1 + \max_{s\in[0,1]} \|Dp_s\|_{r-2})^{r-1}\end{aligned}$$

and

$$\int_0^1 \left\| \frac{d}{ds} D^l p_s \right\| ds \le \ell_{r-1}(p_s)$$

for $0 \le l \le r-1$. Finally

$$\|D^{k+1} h \circ p_s\|, \|D^k h \circ p_s\| \le \|Dh\|_r.$$

Combining these estimates we get the required result. □

Finally we can combine these two to give a more convenient form for the mean value theorem.

Lemma 5.3. *Let $C > 0$ and $r \in \mathbb{N}$ be arbitrary. Suppose $h \in \mathrm{Diff}^r(N)$ and $g_1, g_2 \in \mathrm{Diff}^{r-1}(N)$. There exists a constant $C' > 0$ such that if*

$$d_r(h, \mathrm{Id}) < C, \quad d_{r-1}(g_1, \mathrm{Id}) < C, \quad d_{r-1}(g_2, \mathrm{Id}) < C$$

then

$$d_{r-1}(h \circ g_1, h \circ g_2) < C' \, d_{r-1}(g_1, g_2),$$
$$d_{r-1}(g_1 \circ h, g_2 \circ h) < C' \, d_{r-1}(g_1, g_2).$$

The constant C' depends on C, r, and the manifold N.

Proof. Since $d_{r-1}(g_1, g_2) < 2C$ we may take a path p_s joining g_1 to g_2 with $\ell_{r-1}(p_s) < 2C$ and $\ell_{r-1}(p_s^{-1}) < 2C$. Using Lemma 5.1 we see that there exists $C_1 > 0$ with $\|Dp_s\|_{r-2} < C_1$ and $\|Dp_s^{-1}\|_{r-2} < C_1$. Again by Lemma 5.1 since $d_r(h, \mathrm{Id}) < C$ there exists $C_2 > 0$ such that $\|Dh\|_{r-1} < C_2$. Now by Lemma 5.2 there exists $C_3 > 0$, depending only on r, such that

$$\ell_{r-1}(h \circ p_s) \le C_3 \, \|Dh\|_{r-1} \, (1 + \|Dp_s\|_{r-2})^{r-1} \, \ell_{r-1}(p_s),$$
$$\ell_{r-1}(p_s^{-1} \circ h^{-1}) \le C_3 \left(1 + \|Dh^{-1}\|_{r-2}\right)^{r-1} \ell_{r-1}(p_s^{-1}).$$

All these quantities are bounded so we have a $C' > 0$ such that

$$\ell_{r-1}(h \circ p_s) \le C' \, \ell_{r-1}(p_s),$$
$$\ell_{r-1}(p_s^{-1} \circ h^{-1}) \le C' \, \ell_{r-1}(p_s^{-1}).$$

Taking infimums we obtain

$$d_{r-1}(h \circ g_1, h \circ g_2) \le C' d_{r-1}(g_1, g_2).$$

The other direction is an immediate consequence of the symmetry of our metric. □

5.3. Preliminary Estimates for the Flow Case.

Lemma 5.4. *Let $\eta \in C^{\alpha}(M, \mathfrak{X}^r(N))$ and define*

$$\rho_0 = \max_{x\in M} \max_{y\in N} \|\eta_x(y)\|$$
$$\rho_1 = \max_{x\in M} \max_{y\in N} \|D_y \eta_x\|.$$

Define the cocycle $\Phi : M \times \mathbb{R} \to \mathrm{Diff}^r(N)$ by

$$\frac{d}{ds}\Phi(x,s) = \eta_{f^s x} \circ \Phi(x,s), \qquad \Phi(x,0) = \mathrm{Id}\,.$$

Then we have the following estimates for $n \le r$

$$d_0(\Phi(x,s), \mathrm{Id}) \le \rho_0\,|s|, \qquad \|D^n\Phi(x,s)\| \le C\, e^{n(\rho_1 + \kappa\,\rho_0)|s|}$$

where C is a constant that depends only on r and κ is the geometric constant introduced above.

Proof. To aid in readability we will write Φ_s for $\Phi(x,s)$ since x plays no rôle in this lemma. We immediately have

$$\left\| \frac{d}{ds}\Phi_s(y) \right\|_{\Phi_s(y)} \le \rho_0 \tag{27}$$

which, upon integrating, establishes the first estimate.

We proceed by induction to establish the remaining estimates. For fixed $y \in N$ and $v \in T_y N$ we have

$$\frac{d}{ds} D_y \Phi_s(y)\, v = D_{\Phi_s(y)} \eta_{f^s x} \cdot D_y \Phi_s\, v, \qquad D\Phi(x,0) = \mathrm{Id}$$

and thus

$$\|\frac{d}{ds} D_y \Phi_s\, v\|_{\Phi_s(y)} = \|D_{\Phi_s} \eta_{f^s x}\|\, \|D_y \Phi_s\, v\|_{\Phi_s(y)} \le \rho_1\, \|D_y \Phi_s\, v\|_{\Phi_s(y)} \tag{28}$$

Exactly as in Lemma 5.1, using (25) yields the following estimate,

$$\frac{d}{ds}\|D_y \Phi_s\, v\|_{\Phi_s(y)}$$
$$\le \|\frac{d}{ds} D_y \Phi_s\, v\|_{\Phi_s(y)} + \kappa\, \|D_y \Phi_s\, v\|_{\Phi_s(y)}\, \|\frac{d}{ds}\Phi_s(y)\|_{\Phi_s(y)}.$$

Using (27) and (28) we get

$$\frac{d}{ds}\|D_y \Phi_s\, v\|_{\Phi_s(y)} \le \rho_1 \|D_y \Phi_s\, v\|_{\Phi_s(y)} + \kappa\, \rho_0\, \|D_y \Phi_s\, v\|_{\Phi_s(y)}$$

The Gronwall inequality gives

$$\|D_y \Phi_s\, v\| \le e^{(\rho_1 + \kappa\,\rho_0)|s|}$$

which establishes the base case.

Applying the Faà di Bruno formula to $\eta_{f^s x} \circ \Phi_s$ we obtain

$$\frac{d}{ds} D_y^n [\eta_{f^s x} \circ \Phi_s] = \sum_{k_1,\ldots,k_n} C_{k_1,\ldots,k_n}\, D^k_{\Phi_s(y)} \eta_{f^s x} \cdot \big(D_y \Phi_s\big)^{\otimes k_1} \otimes \cdots \otimes \big(D_y^n \Phi_s\big)^{\otimes k_n}$$

where $k = k_1 + \cdots + k_n$ and the sum is taken over all $k_1, \ldots, k_n$ such that $k_1 + 2k_2 + \cdots + nk_n = n$.

Thus we obtain for any

$$\|\frac{d}{ds} D_y^n \Phi_s\| \leq \sum_{k_1,\ldots,k_n} C_{k_1,\ldots,k_n} \|D^k_{\Phi_s(y)} \eta_{f^s x}\| \, \|D_y \Phi_s\|^{k_1} \cdots \|D_y^n \Phi_s\|^{k_n}$$

Separating this into $k_n = 1$ and k_n=0 terms we obtain

$$\begin{aligned} \|\frac{d}{ds} D_y^n \Phi_s\| \leq & \|D_{\Phi_s(y)} \eta_{f^s x}\| \, \|D_y^n \Phi_s\| \\ & + \sum_{k_n=0} C_{k_1,\ldots,k_{n-1}} \|D^k_{\Phi_s(y)} \eta_{f^s x}\| \, \|D_y \Phi_s\|^{k_1} \cdots \|D_y^{n-1} \Phi_s\|^{k_{n-1}} \\ & \leq \rho_1 \, \|D_y^n \Phi_s\| + C\, e^{n(\rho_1 + \kappa\, \rho_0)|s|}. \end{aligned}$$

Again, as in Lemma 5.1, we obtain for any $v \in (T_y N)^{\otimes n}$

$$\begin{aligned} \frac{d}{ds} \|D_y^n \Phi_s\, v\|_{\Phi_s(y)} &\leq \|\frac{d}{ds} D_y^n \Phi_s\, v\|_{\Phi_s(y)} + \kappa\, \|D_y^n \Phi_s\, v\|_{\Phi_s(y)} \, \|\frac{d}{ds} \Phi_s(y)\|_{\Phi_s(y)} \\ &\leq \rho_1 \, \|D_y^n \Phi_s\, v\| + C\, e^{n(\rho_1 + \kappa\, \rho_0)|s|} + \kappa\, \rho_0 \, \|D_y^n \Phi_s\, v\| \end{aligned}$$

Using a Gronwall-type inequality we then obtain

$$\|D_y^n \Phi_s\| \leq C\, e^{n(\rho_1 + \kappa\, \rho_0)|s|}$$

as required. □

5.4. **Preliminary Estimates for the Diffeomorphism Case.**

Lemma 5.5. *Let $\eta \in C^\alpha(M, \mathrm{Diff}^r(N)$ and define*

$$\begin{aligned} \rho_0 &= \max_{x \in M} d_0(\eta_x, \mathrm{Id}) \\ \rho_1 &= \max_{x \in M} \max\{\|D\eta_x\|, \|D\eta_x^{-1}\|\}. \end{aligned}$$

Define the cocycle $\Phi : M \times \mathbb{Z} \to \mathrm{Diff}^r(N)$ by

$$\Phi(x, n) = \begin{cases} \eta_{f^{n-1}x} \circ \cdots \circ \eta_x & n \geq 1 \\ \mathrm{Id} & n = 0 \\ \eta^{-1}_{f^n x} \circ \cdots \circ \eta^{-1}_{f^{-1}x} & n \leq -1 \end{cases}.$$

The we have the following estimates for $m \leq r$

$$d_0(\Phi(x, n), \mathrm{Id}) \leq \rho_0 |n| \qquad \|D^m \Phi(x, n)\| \leq C \rho_1^{m\,|n|}.$$

Proof. By the triangle inequality we have for $n \geq 1$

$$\begin{aligned} d_0(\Phi(x,n),\mathrm{Id}) &\leq d_0\big(\Phi(x,n),\Phi(x,n-1)\big) + \cdots + d_0\big(\Phi(x,1),\mathrm{Id}\big) \\ &\leq d_0\big(\eta_{f^{n-1}x}\circ\Phi(x,n-1),\Phi(x,n-1)\big) + \cdots + d_0\big(\eta_x,\mathrm{Id}\big) \\ &\leq d_0\big(\eta_{f^{n-1}x},\mathrm{Id}\big) + \cdots + d_0\big(\eta_x,\mathrm{Id}\big) \\ &\leq n \max_{x\in M} d_0(\eta_x,\mathrm{Id}). \end{aligned}$$

Since $d_0(\eta_x,\mathrm{Id}) = d_0(\eta_x^{-1},\mathrm{Id})$ a similar argument works for $n \leq 0$ too.

We have the following basic evolution equation

$$\Phi(x,n+1) = \eta_{f^n x}\circ\Phi(x,n)$$

and hence we have

$$D\Phi(x,n+1) = D\eta_{f^n x}\circ\Phi(x,n)\cdot D\Phi(x,n).$$

Taking norms, we get

$$\begin{aligned} \|D\Phi(x,n+1)\| &\leq \|D\eta_{f^n x}\|\,\|D\Phi(x,n)\| \\ &\leq \rho_1\,\|D\Phi(x,n)\|. \end{aligned}$$

Finally since $\Phi(x,0)=\mathrm{Id}$, and hence $\|D\Phi(x,0)\|=1$, we see that we have

$$\|D\Phi(x,n)\| \leq \rho_1^n$$

for all $n \geq 0$. Observe that for $n \leq 0$ we have the following evolution equation

$$\Phi(x,n-1) = \eta^{-1}_{f^{n-1}x}\circ\Phi(x,n)$$

so exactly as above we obtain

$$\begin{aligned} \|D\Phi(x,n-1)\| &\leq \|D\eta^{-1}_{f^{n-1}x}\|\,\|D\Phi(x,n)\| \\ &\leq \rho_1\,\|D\Phi(x,n)\|. \end{aligned}$$

Hence we get

$$\|D\Phi(x,n)\| \leq \rho_1^{|n|}$$

for $n \leq 0$.

We now proceed by induction. Suppose that we have the estimate

$$\|D^k\Phi(x,n)\| \leq C\,\rho_1^{k\,n}$$

for all $k < m$. We will now establish the estimate for m. Applying the Faà di Bruno formula to our basic evolution equation we obtain

$$\begin{aligned} D^m\Phi(x,n+1) = \sum_{k_1,\cdots,k_m} & D^k\eta_{f^n x}\circ\Phi(x,n) \\ & \cdot\Big[\big(D^1\Phi(x,n)\big)^{\otimes k_1}\otimes\cdots\otimes\big(D^m\Phi(x,n)\big)^{\otimes k_m}\Big] \end{aligned}$$

where $k = k_1 + \cdots k_m$ and the sum is taken over all $k_1, \ldots, k_m \geq 0$ such that $k_1 + 2k_2 + \cdots + mk_m = m$. Either $k_m = 0$ or $k_m = 1$. We separate these terms

$$\begin{aligned} &D^m\Phi(x, n+1) = \\ &\qquad D^1\eta_{f^n x} \circ \Phi(x,n) \cdot D^m\Phi(x,n) + \sum_{k_1,\cdots,k_{m-1}} D^k\eta_{f^n x} \circ \Phi(x,n) \\ &\qquad\qquad \cdot \Big[\big(D^1\Phi(x,n)\big)^{\otimes k_1} \otimes \cdots \otimes \big(D^{m-1}\Phi(x,n)\big)^{\otimes k_{m-1}}\Big]. \end{aligned}$$

Taking norms we obtain

$$\begin{aligned} &\|D^m\Phi(x,n+1)\| \leq \|D^1\eta_{f^n x}\| \, \|D^m\Phi(x,n)\| \\ &\qquad + \sum_{k_1,\cdots,k_{m-1}} \|D^k\eta_{f^n x}\| \|D^1\Phi(x,n)\|^{k_1} \cdots \|D^{m-1}\Phi(x,n)\|^{k_{m-1}} \end{aligned}$$

Now applying the inductive assumption we obtain

$$\|D^m\Phi(x,n+1)\| \leq \rho_1 \, \|D^m\Phi(x,n)\| + C\,\rho_1^{mn}$$

where C depends on m and on η. Using this we can check that if

$$\|D^m\Phi(x,n)\| \leq \frac{C}{\rho_1^m - \rho_1}\rho_1^{m\,n}$$

then

$$\|D^m\Phi(x,n+1)\| \leq \frac{C}{\rho_1^m - \rho_1}\rho_1^{m\,(n+1)}.$$

Since the estimate is obviously true for $n = 0$ we have established it for all $n \geq 0$. Starting with the evolution equation

$$\Phi(x, n-1) = \eta^{-1}_{f^{n-1}x} \circ \Phi(x,n)$$

and applying the same estimates establishes the result for all $n \leq 0$.

□

5.5. **Main Theorem for Diffeomorphism Valued Cocycles.** Let $\mathfrak{X}^r(N)$ denote the space of C^r vector fields on N.

Theorem 5.6. *Let M be a compact Riemannian manifold with $f^t : M \to M$ be a C^1 topologically transitive λ-hyperbolic Anosov flow. Let N be a compact Riemannian manifold.*

Given a $\eta \in C^\alpha\big(M, \mathfrak{X}^r(N)\big)$ define a cocycle $\Phi : M \times \mathbb{R} \to \mathrm{Diff}^r(N)$ by

$$\frac{d}{dt}\Phi(x,t) = \eta(f^t x) \circ \Phi(x,t), \qquad \Phi(x,0) = \mathrm{Id}\,.$$

Let

$$\rho_0 = \max_{x\in M}\max_{y\in N}\|\eta_x(y)\|$$
$$\rho_1 = \max_{x\in M}\max_{y\in N}\|D_y\eta_x\|.$$

Suppose that for the pair f^t and η:

(i) *The periodic orbit obstruction vanishes:*
If $f^t p = p$ then $\Phi(p,t) = \mathrm{Id}$.

(ii) *The hyperbolicity condition is satisfied:*

$$(2r-1)(\rho_1 + \kappa\rho_0) - \lambda\alpha < 0. \tag{29}$$

Then there exists $\phi \in C^\alpha\big(M, \mathrm{Diff}^{r-3}(N)\big)$ that solves

$$\Phi(x,t) = \phi(f^t x)\circ\phi^{-1}(x). \tag{30}$$

Remark 5.7. *Using Hölder estimates it should be possible to show that the solution $\phi \in \mathrm{Diff}^{r-\epsilon}(N)$* [dlLO99]. *Using different arguments we hope to be able to show that $\phi \in \mathrm{Diff}^r(N)$ so that there is no loss of differentiability.*

Proof. Let $x^* \in M$ be a point with a dense orbit, $\mathcal{O}(x^*)$. If we fix $\phi(x^*)$ then, by (30), we can define ϕ on all of $\mathcal{O}(x^*)$ by

$$\phi(f^t x^*) = \Phi(x^*, t)\circ\phi(x^*).$$

First, we show that the following Hölder condition on Φ,

$$\text{if } d_M(f^{t+T}x^*, f^t x^*) < \delta \text{ then } \quad d_{r-2}\big(\Phi(f^t x^*, T), \mathrm{Id}\big) < K\, d_M(f^{t+T}x^*, f^t x^*)^\alpha \tag{31}$$

implies the following Hölder condition

$$\text{if } d_M(f^{t+T}x^*, f^t x^*) < \delta \text{ then } \quad d_{r-3}\big(\phi(f^{t+T}x^*), \phi(f^t x^*)\big) < K\, d_M(f^{t+T}x^*, f^t x^*)^\alpha. \tag{32}$$

Condition (32) is precisely the condition that ensures that ϕ defined on $\mathcal{O}(x^*)$ can be extended to $\phi \in C^\alpha\big(M, \mathrm{Diff}^{r-3}(N)\big)$.

The collection $\{B(f^t x^*, \delta)\}_{t\in\mathbb{R}}$ is an open cover of M and therefore by compactness we have a finite sub-cover $\{B(f^{t_i}x^*, \delta)\}_{i=1}^m$. By finiteness there exists a constant $C > 0$ such that $d_r(\phi(f^{t_i}x^*), \mathrm{Id}) < C$ for $1 \le i \le m$. Given an arbitrary $t \in \mathbb{R}$ we choose $1 \le i \le m$ such that $d_M(f^t x^*, f^{t_i}x^*) < \delta$. From (31) we get $d_{r-2}\big(\Phi(f^{t_i}x^*, t - t_i), \mathrm{Id}\big) < K\,\delta^\alpha$. By Lemma 5.3 we have

$$\begin{aligned} d_{r-2}\big(\Phi(f^{t_i}x^*, t-t_i)\circ\phi(f^{t_i}x^*), \phi(f^{t_i}x^*)\big) \\ \le C\, d_{r-2}\big(\Phi(f^{t_i}x^*, t-t_i), \mathrm{Id}\big) \le C\,K\,\delta^\alpha. \end{aligned}$$

In particular $d_{r-2}\big(\phi(f^t x^*), \phi(f^{t_i} x^*\big)$ is bounded. However since $d_{r-2}\big(\phi(f^{t_i} x^*), \mathrm{Id}\big)$ is bounded we have that $d_{r-2}\big(\phi(f^t x^*), \mathrm{Id}\big)$ is bounded.

Suppose that t and T are such that $d_M(f^t x*, f^{t+T} x^*) < \delta$. Assuming (31) we then have $d_{r-2}\big(\Phi(f^t x^*, T), \mathrm{Id}\big)$ uniformly bounded. We can again apply Lemma 5.3 to obtain

$$d_{r-3}\big(\phi(f^t x^*), \phi(f^{t+T} x^*)\big) \leq C\, d_{r-3}\big(\Phi(f^t x^*, T), \mathrm{Id}\big).$$

Then applying (31) we obtain (32). Therefore to prove the theorem it suffices to prove (31).

Suppose that $d_M(f^{t+T} x^*, f^t x^*) < \delta$ and apply the Anosov Closing Lemma, Lemma 2.4, to obtain a periodic point $p \in M$ with $f^{T+\Delta} p = p$ and a point $z \in W^s(p) \cap W^u(f^t x^*)$. The periodic point satisfies:

(i) $|\Delta| < K\, d_M(f^{t+T} x^*, f^t x^*)$.
(ii) $d_M(f^{t+T} x^*, p) \leq K\, d_M(f^{t+T} x^*, f^t x^*)$.

Again we let $C_F(s) := \Phi^{-1}(p, s) \circ \Phi(z, s)$. Here we encounter our first difficulty; even though $\Phi(p, s)$ and $\Phi(z, s)$ are differentiable as maps from $\mathbb{R}$ to $\mathrm{Diff}^r(N)$ the map $C_F(s)$ is not since the group operation is not differentiable. The solution is to consider $C_F(s)$ in $\mathrm{Diff}^{r-1}(N)$. The diffeomorphism $C_F(s)$ obeys the following differential equation in $T\,\mathrm{Diff}^{r-1}(N)$,

$$\begin{aligned} \frac{d}{ds} C_F(s) &= D\Phi^{-1}(p, s) \circ \Phi(z, s) \cdot [\eta_{f^s z} \circ \Phi(z, s) - \eta_{f^s p} \circ \Phi(z, s)], \\ C_F(0) &= \mathrm{Id}\,. \end{aligned}$$

This equation is exactly analogous to the equation obtained in the Lie group case (21). However, rather than deal with the Banach manifold $T\,\mathrm{Diff}^{r-1}(N)$ we wish to use the familiar theory of differential equations on the compact manifold N.

We wish to estimate $C_F(s)$ in C^{r-1}. Since $\Phi^{-1}(p, s) = \Phi(f^s p, -s)$ we can estimate $D\Phi^{-1}(p, s)$ and $\Phi(z, s)$ using Lemma 5.4. The fact that $\eta \in C^\alpha(M, \mathrm{Diff}^r(N))$ means that we have

$$\|D^n \eta_{f^s z} - D^n \eta_{f^s p}\| \leq C\, d_M(f^s z, f^s p)^\alpha \tag{33}$$

for $0 \leq n \leq r$. Since $z \in W^s(p)$ we have

$$d_M(f^s z, f^s p) < e^{-\lambda s}\, d_M(z, p).$$

From our statement of the Anosov Closing Lemma, Lemma 2.4, we have

$$d_M(z, p) \leq C d_M(f^t x^*, f^{t+T} x^*).$$

Using Lemma 5.4, and (33), we estimate

$$\begin{aligned}\|D\Phi^{-1}(p,s)\circ\Phi(z,s)\cdot[\eta_{f^s z}\circ\Phi(z,s)-\eta_{f^s p}\circ\Phi(z,s)]\|&\\ \le \|D\Phi^{-1}(p,s)\|\,\|\eta_{f^s z}-\eta_{f^s p}\|&\\ \le C\,e^{(\rho_1+\kappa\,\rho_0-\lambda\alpha)s}\,d_M(f^t x^*, f^{t+T}x^*)^\alpha&\end{aligned}$$

and

$$\begin{aligned}\|D\Phi^{-1}(z,s)\circ\Phi(p,s)\cdot[\eta_{f^s p}\circ\Phi(p,s)-\eta_{f^s z}\circ\Phi(p,s)]\|&\\ \le \|D\Phi^{-1}(z,s)\|\,\|\eta_{f^s p}-\eta_{f^s z}\|&\\ \le C\,e^{(\rho_1+\kappa\,\rho_0-\lambda\alpha)s}\,d_M(f^t x^*, f^{t+T}x^*)^\alpha.&\end{aligned}$$

Integrating these estimates, and using (29), we obtain

$$d\big(C_F(s),\mathrm{Id}\big)\le\frac{C}{\lambda\alpha-\rho}d_M(f^t x^*, f^{t+T}x^*)^\alpha.$$

It remains to estimate the derivatives $D^nC_F(s)$ and $D^nC_F^{-1}(s)$ for $n\le r-1$. Applying the Faà di Bruno formula to the differential equation for $D^nC_F(s)$ we obtain

$$\begin{aligned}\frac{d}{ds}D^nC_F(s)&=D^n\Big(\big(D\Phi^{-1}(p,s)\cdot[\eta_{f^s z}-\eta_{f^s p}]\big)\circ\Phi(z,s)\Big)\\ &=\sum_{k_1,\dots,k_n}C_{k_1,\dots,k_n}D^k\big(D\Phi^{-1}(p,s)\cdot[\eta_{f^s z}-\eta_{f^s p}]\big)\\ &\qquad\cdot(D\Phi(z,s))^{\otimes k_1}\cdots(D^n\Phi(z,s))^{\otimes k_n}\end{aligned}$$

The general term consists of products of two types of factors:

$$D^n\Phi(z,s)\text{ and }D^n\big(D\Phi^{-1}(p,s)\cdot[\eta_{f^s z}-\eta_{f^s p}]\big).$$

The term $D^n\Phi(z,s)$ is estimated using Lemma 5.4. To estimate

$$\|D^n\big[D\Phi^{-1}(p,s)\cdot[\eta_{f^s z}-\eta_{f^s p}]\big]\|$$

we apply the Leibniz rule to obtain

$$D^n\big[D\Phi^{-1}(p,s)\cdot[\eta_{f^s z}-\eta_{f^s p}]\big]=\sum_{k=0}^{n}D^{k+1}\Phi^{-1}(p,s)\cdot[D^{n-k}\eta_{f^s z}-D^{n-k}\eta_{f^s p}].$$

As above, we obtain

$$\begin{aligned}\|D^n\big[D\Phi^{-1}(p,s)\cdot[\eta_{f^s z}-\eta_{f^s p}]\big]\|&\\ \le C\,e^{(n+1)(\rho_1+\kappa\,\rho_0)s}d_M(f^s z, f^s p)^\alpha&\\ \le C\,e^{(n+1)(\rho_1+\kappa\,\rho_0)s-\lambda\alpha s}d_M(z,p)^\alpha&\\ \le C\,e^{(n+1)(\rho_1+\kappa\,\rho_0)s-\lambda\alpha s}d_M(f^t x^*, f^{t+T}x^*)^\alpha&\end{aligned}$$

Using the estimate from Lemma 5.4, and our intermediate computation, we obtain

$$\Big\|\frac{d}{ds}D^n C_F(s)\Big\| \le C\, e^{((2n+1)(\rho_1+\kappa\,\rho_0)-\lambda\alpha)s}\, d_M(f^t x^*, f^{t+T}x^*)^\alpha.$$

Similarly, we get

$$\Big\|\frac{d}{ds}D^n C_F^{-1}(s)\Big\| \le C\, e^{((2n+1)(\rho_1+\kappa\,\rho_0)-\lambda\alpha)s}\, d_M(f^t x^*, f^{t+T}x^*)^\alpha.$$

Integrating, and using (29), we obtain the following estimate

$$d_{r-1}\big(C_F(s),\mathrm{Id}\big) \le \frac{C}{\lambda\alpha-(2r-1)(\rho_1+\kappa\,\rho_0)}\, d_M(f^t x^*, f^{t+T}x^*)^\alpha.$$

Now we can write

$$\begin{aligned} d_{r-1}\big(\Phi(z,T),\mathrm{Id}\big) \le\ & d_{r-1}\big(\Phi(p,T)\circ\Phi^{-1}(p,T)\circ\Phi(z,T),\Phi(p,T)\big)\\ &+ d_{r-1}\big(\Phi(p,T),\mathrm{Id}\big)\end{aligned}$$

From the cocycle property and the periodic orbit obstruction we have $\Phi(p,T)=\Phi(p,-\Delta)$. From the Anosov Closing Lemma we have $|\Delta| < C\, d_M(f^t x^*, f^{t+T}x^*)$ and hence Δ is bounded. Thus we immediately get

$$d_{r-1}\big(\Phi(p,T),\mathrm{Id}\big) \le C\,|\Delta|.$$

Consider $C_F(s)$ for $0\le s\le T$ as a path. Using Lemma 5.3, we estimate

$$\begin{aligned}&\ell_{r-1}\big(\Phi(p,T)\circ C_F(s)\big)\\ &\qquad \le C\,\|D\Phi(p,T)\|_{r-1}\,\big(1+\max_{s\in[0,T]}\|DC_F(s)\|_{r-2}\big)^{r-1}\,\ell_{r-1}\big(C_F(s)\big)\end{aligned}$$

Since $\Phi(p,T+\Delta)=\mathrm{Id}$ and Δ is bounded we have $\|D\Phi(p,T)\|_{r-1}$ is uniformly bounded. Since $d_{r-1}\big(C_F(s),\mathrm{Id}\big)$ is uniformly bounded we have by Lemma 5.1 that $\max_{s\in[0,T]}\|DC_F(s)\|_{r-2}$ is uniformly bounded. Similarly, using Lemma 5.3, we estimate

$$\ell_{r-1}\big(C_F^{-1}(s)\circ\Phi^{-1}(p,T)\big) \le C\,(1+\|D\Phi^{-1}(p,T)\|_{r-2})^{r-1}\ell_{r-1}\big(C_F^{-1}(s)\big).$$

The term $\|D\Phi^{-1}(p,T)\|_{r-2}$ is uniformly bounded since Δ is bounded. Thus we finally obtain

$$\begin{aligned} &d_{r-1}\big(\Phi(p,T)\circ\Phi^{-1}(p,T)\circ\Phi(z,T),\Phi(p,T)\big)\\ &\qquad\le C\, d_{r-1}\big(C_F(T),\mathrm{Id}\big)\\ &\qquad\le \frac{C}{\lambda\alpha-(2r-1)(\rho_1+\kappa\,\rho_0)}\, d_M(f^t x^*, f^{t+T}x^*)^\alpha\end{aligned}$$

Combining these estimates, we get

$$d_{r-1}\big(\Phi(z,T),\mathrm{Id}\big) \le C\, d_M(f^t x^*, f^{t+T}x^*)^\alpha \tag{34}$$

Now we compare the cocycle along orbits that converge in backward time. Let

$$C_R(s) = \Phi^{-1}(f^{t+T}x^*, -s) \circ \Phi(f^T z, -s).$$

The function $C_R(s)$ satisfies the differential equation

$$\frac{d}{ds} C_R(s) = \big(D\Phi^{-1}(f^{t+T}x^*, -s)\cdot[\eta_{f^{t+T-s}x^*} - \eta_{f^{T-s}z}]\big) \circ \Phi(f^T z, -s). \tag{35}$$

Differentiating the differential equation (35) n times, we obtain the differential equation

$$\frac{d}{ds} D^n C_R(s) = D^n\Big(\big(D\Phi^{-1}(f^{t+T}x^*, -s)\cdot[\eta_{f^{t+T-s}x^*} - \eta_{f^{T-s}z}]\big) \circ \Phi(f^T z, -s)\Big).$$

Now we apply the Faà di Bruno formula to the differential equation for $D^n C_R(s)$ to obtain

$$\begin{aligned}\frac{d}{ds} D^n C_R(s) &= D^n\Big(\big(D\Phi^{-1}(f^{t+T}x^*, -s)\cdot[\eta_{f^{t+T-s}x^*} - \eta_{f^{T-s}z}]\big) \circ \Phi(f^T z, -s)\Big) \\ &= \sum C_{k_1,\dots,k_n} D^k\big(D\Phi^{-1}(f^{t+T}x^*, -s)\cdot[\eta_{f^{t+T-s}x^*} - \eta_{f^{T-s}z}]\big) \\ &\qquad \cdot \big[D\Phi(f^T z, -s)^{\otimes k_1} \otimes \cdots \otimes D^n\Phi(f^T z, -s)^{\otimes k_n}\big]\end{aligned}$$

The general term in the sum consists of the product of two types of factors:

$$D^k\big(D\Phi^{-1}(f^{t+T}x^*, -s)\cdot[\eta_{f^{t+T-s}x^*} - \eta_{f^{T-s}z}]\big) \text{ and } D^k\Phi(f^T z, -s).$$

Taking the n-th derivative

$$\begin{aligned}&D^n\big(D\Phi^{-1}(f^{t+T}x^*, -s)\cdot[\eta_{f^{t+T-s}x^*} - \eta_{f^{T-s}z}]\big) \\ &\quad = \sum_{k=0}^{n} D^{k+1}\Phi^{-1}(f^{t+T}x^*, -s)\cdot[D^{n-k}\eta_{f^{t+T-s}x^*} - D^{n-k}\eta_{f^{T-s}z}]\end{aligned}$$

which we may estimate by

$$\begin{aligned}&\big\|D^n\big(D\Phi^{-1}(f^{t+T}x^*, -s)\cdot[\eta_{f^{t+T-s}x^*} - \eta_{f^{T-s}z}]\big)\big\| \\ &\qquad \le C\, e^{(n+1)(\rho_1+\kappa\,\rho_0)s} d_M(f^{T-s}z, f^{t+T-s}x^*)^\alpha \\ &\qquad \le C\, e^{((n+1)(\rho_1+\kappa\,\rho_0)-\lambda\alpha)s}\, d_M(f^T z, f^{t+T}x^*)^\alpha \\ &\qquad \le C\, e^{((n+1)(\rho_1+\kappa\,\rho_0)-\lambda\alpha)s}\, d_M(f^t x^*, f^{t+T}x^*)^\alpha\end{aligned}$$

Using the estimate from Lemma 5.4, and our intermediate computation, we obtain

$$\Big\|\frac{d}{ds} D^n C_R(s)\Big\| \le C e^{((2n+1)(\rho_1+\kappa\,\rho_0)-\lambda\alpha)s}\, d_M(f^t x^*, f^{t+T}x^*)^\alpha.$$

Integrating, and using (29), we obtain

$$d_{r-1}\big(C_R(T), \mathrm{Id}\big) \le \frac{C}{\lambda\alpha - (2r-1)(\rho_1 + \kappa\,\rho_0)}\, d_M(f^t x^*, f^{t+T} x^*)^\alpha. \tag{36}$$

Finally we need to combine these estimates to obtain the final result. First observe that from the cocycle condition

$$\begin{aligned} C_R(T) &= \Phi^{-1}(f^{t+T}x^*, -T) \circ \Phi(f^T z, -T) \\ &= \Phi(f^t x^*, T) \circ \Phi^{-1}(z, T) \end{aligned}$$

Now

$$\begin{aligned} d_{r-1}\big(\Phi(f^t x^*, T), \mathrm{Id}\big) \le\ & d_{r-1}\big(\Phi(f^t x^*, T) \circ \Phi^{-1}(z,T) \circ \Phi(z,T), \Phi(z,T)\big) \\ & + d_{r-1}\big(\Phi(z,T), \mathrm{Id}\big) \end{aligned}$$

Using Lemma 5.3, and (34), gives

$$d_{r-2}\big(\Phi(f^t x^*, T) \circ \Phi^{-1}(z,T) \circ \Phi(z,T), \Phi(z,T)\big) \le C\, d_{r-1}\big(C_R(T), \mathrm{Id}\big).$$

Combining all our estimates yields

$$d_{r-2}\big(\Phi(f^t x^*, T), \mathrm{Id}\big) \le C\, d_M(f^t x^*, f^{t+T} x^*)^\alpha \tag{37}$$

which hence completes the proof. □

5.6. **Cocycles over an Anosov Diffeomorphism.** A statement analogous to Theorem 5.6 holds for diffeomorphism group valued cocycles over an Anosov diffeomorphism. This can be obtained from the result on flows by observing that the suspension of an Anosov diffeomorphism is an Anosov flow. Proceeding in this fashion one needs to take a cocycle whose generator is very close to the identity.

Theorem 5.8. *Let M be a compact Riemannian manifold with $f : M \to M$ be a C^1 topologically transitive λ-hyperbolic Anosov diffeomorphism. Let N be a compact Riemannian manifold.*

Let $\Phi \in C^\alpha\big(M \times \mathbb{Z}, \mathrm{Diff}^r(N)\big)$ and define $\eta(x) = \Phi(x,1)$. $\rho = \max_{x\in M} \|D\eta(x)\|$.

Suppose that for the pair f^t and η:

(i) *The periodic orbit obstruction vanishes:*
If $f^n p = p$ then $\Phi(p,n) = \mathrm{Id}$.

(ii) *The hyperbolicity condition is satisfied:*

$$\rho^{2r-1}\lambda^\alpha < 1. \tag{38}$$

Then there exists $\phi \in C^\alpha\big(M, \mathrm{Diff}^{r-3}(N)\big)$ that solves

$$\Phi(x,n) = \phi(f^n x) \circ \phi^{-1}(x). \tag{39}$$

Proof. Let $x^* \in M$ be a point with a dense orbit, $\mathcal{O}(x^*)$. If we fix $\phi(x^*)$ then, by (39), we can define ϕ on all of $\mathcal{O}(x^*)$ by

$$\phi(f^n x^*) = \Phi(x^*, n) \circ \phi(x^*).$$

Exactly as in the flow case, we have that the following Hölder condition on Φ,

$$\text{if } d_M(f^{n+N}x^*, f^n x^*) < \delta \text{ then} \\ d_{r-2}\big(\Phi(f^n x^*, N), \mathrm{Id}\big) < K\, d_M(f^{n+N}x^*, f^n x^*)^\alpha \tag{40}$$

implies the Hölder condition

$$\text{if } d_M(f^{n+N}x^*, f^n x^*) < \delta \text{ then} \\ d_{r-3}\big(\phi(f^{n+N}x^*), \phi(f^n x^*)\big) < K\, d_M(f^{n+N}x^*, f^n x^*)^\alpha. \tag{41}$$

Condition (41) means that ϕ can be extended to $\phi \in C^\alpha\Big(M, \mathrm{Diff}^{r-3}(N)\Big)$.

In order to complete the proof it suffices to prove (40). Suppose that $d_M(f^{n+N}x^*, f^n x^*) < \delta$ and apply the Anosov Closing Lemma, Lemma 2.4, to obtain a periodic point $p \in M$ with $f^N p = p$ and a point $z \in W^s(p) \cap W^u(f^n x^*)$. The periodic point satisfies $d_M(f^{n+N}x^*, p) \le K\, d_M(f^{n+N}x^*, f^n x^*)$ and the messenger point satisfies $d_M(f^{n+N}x^*, z) \le K\, d_M(f^{n+N}x^*, f^n x^*)$.

Again we let $C_F(m) := \Phi^{-1}(p, m) \circ \Phi(z, m)$. The diffeomorphism obeys the following equation

$$C_F(m+1) = \Phi^{-1}(p, m) \circ \eta^{-1}_{f^m p} \circ \eta_{f^m z} \circ \Phi(p, m).$$

Let p_s be a path joining $\eta^{-1}_{f^m p} \circ \eta_{f^m z}$ to Id. Since we are only interested in paths that approach the optimal, and since $d_{r-1}(\eta_x^{-1} \circ \eta_{x'}, \mathrm{Id})$ is uniformly bounded, we may assume that $\ell_{r-1}(p_s)$ and $\ell_{r-1}(p_s^{-1})$ are uniformly bounded. Now using Lemma 5.2 we have

$$\ell_{r-1}(\Phi^{-1}(p,m) \circ p_s) \le C \|D\Phi^{-1}(p,m)\|_{r-1} (1 + \max_{s \in [0,1]} \|Dp_s\|_{r-2})^{r-1}\, \ell_{r-1}(p_s)$$

Using Lemma 5.1 and Lemma 5.5 we obtain

$$\ell_{r-1}(\Phi^{-1}(p,m) \circ p_s) \le C\, \rho^{r\,m}\, \ell_{r-1}(p_s)$$

where C is independent of m. Applying Lemma 5.3 we get

$$\begin{aligned} &\ell_{r-1}(\Phi^{-1}(p,m) \circ p_s \circ \Phi(z,m)) \\ &\qquad \le C\, \rho^{r\,m} \max_{k_1, \dots, k_{r-1}} \|D^1 \Phi(z,m)\|^{k_1} \cdots \|D^{r-1}\Phi(z,m)\|^{k_r}\, \ell_{r-1}(p_s) \end{aligned}$$

which, after applying Lemma 5.5, yields

$$\ell_{r-1}(\Phi^{-1}(p,m) \circ p_s \circ \Phi(z,m)) \le C\, \rho^{r\,m}\, \rho^{(r-1)\,m}\, \ell_{r-1}(p_s).$$

By symmetry we get the same estimate for the inverse. Thus we have

$$d_{r-1}\big(C_F(m+1), C_F(m)\big) \le C\,\rho^{(2r-1)\,m}\, d_{r-1}(\eta_{f^m p}^{-1} \circ \eta_{f^m z}, \mathrm{Id}).$$

Notice that by compactness there exists a $C > 0$ so that $d_r(\eta_x, \mathrm{Id}) < C$ for all $x \in M$. Thus by Lemma 5.3 there exists a $K > 1$ so that

$$\frac{1}{K}\, d_{r-1}(\eta_x, \eta_{x'}) \le d_{r-1}(\eta_x \circ \eta_{x'}^{-1}, \mathrm{Id}) \le K\, d_{r-1}(\eta_x, \eta_{x'}),$$
$$\frac{1}{K}\, d_{r-1}(\eta_x, \eta_{x'}) \le d_{r-1}(\eta_x^{-1} \circ \eta_{x'}, \mathrm{Id}) \le K\, d_{r-1}(\eta_x, \eta_{x'}).$$

Since $\eta \in C^\alpha\big(M, \mathrm{Diff}^r(N)\big)$ and $z \in W^s(p)$ we have

$$d_{r-1}\big(C_F(m+1), C_F(m)\big) \le C\,\rho^{(2r-1)\,m}\,\lambda^{\alpha\,m}\, d_M(p,z)^\alpha.$$

In particular

$$d_{r-1}\big(C_F(m+1), C_F(m)\big) \le C\,\rho^{(2r-1)\,m}\,\lambda^{\alpha\,m}\, d_M(f^{n+N}x^*, f^n x^*)^\alpha.$$

Thus for $m \ge 0$ we have the estimate

$$d_{r-1}\big(C_F(m), \mathrm{Id}\big) < \frac{C}{1-\rho^{2r-1}\lambda^\alpha}\, d_M(f^{n+N}x^*, f^n x^*)^\alpha.$$

Finally we observe that $C_F(N) = \Phi(z,N)$ since $\Phi(p,N) = \mathrm{Id}$ by the vanishing of the periodic orbit obstruction. Thus

$$d_{r-1}\big(\Phi(z,N), \mathrm{Id}\big) < \frac{C}{1-\rho^{2r-1}\lambda^\alpha}\, d_M(f^{n+N}x^*, f^n x^*)^\alpha,$$

and hence $\Phi(z,N)$ is uniformly bounded.

Similar computations for $C_R(m) = \Phi^{-1}(f^{n+N}x^*, -m) \circ \Phi(f^N z, -m)$ give the same result that for $m \ge 0$

$$d_{r-1}\big(C_R(m), \mathrm{Id}\big) < \frac{C}{1-\rho^{2r-1}\lambda^\alpha}\, d_M(f^{n+N}x^*, f^n x^*)^\alpha.$$

Finally we observe that by the triangle inequality

$$\begin{aligned} &d_{r-1}\big(\Phi(f^n x^*, N), \mathrm{Id}\big) \\ &\qquad \le d_{r-1}\big(\Phi(f^n x^*, N) \circ \Phi^{-1}(z,N) \circ \Phi(z,N), \Phi(z,N)\big) \\ &\qquad\qquad + d_{r-1}\big(\Phi(z,N), \mathrm{Id}\big). \end{aligned}$$

Since $d_{r-1}\big(\Phi(z,N), \mathrm{Id}\big)$ is uniformly bounded by (5.6) we get

$$\begin{aligned} &d_{r-2}\big(\Phi(f^n x^*, N), \mathrm{Id}\big) \\ &\qquad \le C\, d_{r-1}\big(\Phi(f^n x^*, N) \circ \Phi^{-1}(z,N), \mathrm{Id}\big) \\ &\qquad\qquad + d_{r-2}\big(\Phi(z,N), \mathrm{Id}\big). \end{aligned}$$

Observe that

$$\Phi(f^n x^*, N) = \Phi^{-1}(f^{n+N} x^*, -N) \text{ and } \Phi^{-1}(z, N) = \Phi(f^N z, -N)$$

so

$$d_{r-1}\big(\Phi(f^n x^*, N) \circ \Phi^{-1}(z, N), \mathrm{Id}\big) = d_{r-1}\big(C_R(N), \mathrm{Id}\big).$$

Combining (5.6) and (5.6) with our previous estimate we get

$$d_{r-2}\big(\Phi(f^n x^*, N), \mathrm{Id}\big) < \frac{C}{1 - \rho^{2r-1}\lambda^\alpha}\, d_M(f^{n+N} x^*, f^n x^*)^\alpha$$

which hence completes the proof. □

An alternative proof by suspension is also possible though the smallness conditions are much less explicit.

Proof. Let $\tilde{M}$ denote the usual suspension manifold

$$\tilde{M} = \frac{M \times [0,1]}{\sim} \qquad (x, 1) \sim (f(x), 0) \tag{42}$$

and define a flow $\tilde{f}^t : \tilde{M} \to \tilde{M}$ by $\tilde{f}^t(x, s) = (x, s + t)$. This flow is a C^1 topologically transitive λ-hyperbolic Anosov flow. It remains to show that we may select $\tilde{\eta} : \tilde{M} \to \mathfrak{X}^r(N)$ such that the cocycle $\tilde{\Phi} : \tilde{M} \times \mathbb{R} \to \mathrm{Diff}^r(N)$ defined by

$$\frac{d}{dt}\tilde{\Phi}\big((x, s), t\big) = \tilde{\eta}\big(\tilde{f}^t(x, s)\big) \circ \tilde{\Phi}\big((x, s), t\big), \qquad \tilde{\Phi}\big((x, s), 0\big) = \mathrm{Id}$$

satisfies $\tilde{\Phi}\big((x, 0), 1\big) = \eta(x)$ and that the hyperbolicity condition for $\tilde{\Phi}$ is the same as for η. Provided $\eta(x)$ is sufficiently C^1 close to the identity then we can define $p : M \times [0, 1] \to \mathrm{Diff}^r(N)$ by

$$p_{(x,s)}(y) = \exp_y\big[m(s) \exp_y^{-1} \eta_x(y)\big]$$

where $m \in C^\infty\big([0, 1], [0, 1]\big)$ is C^∞ flat at both $s = 0$ and $s = 1$, and has $0 \leq m'(s) \leq 1 + \epsilon$. We have $p_{(x,0)} = \mathrm{Id}$ and $p_{(x,1)} = \eta_x$. We may differentiate to obtain

$$\frac{d}{ds} p_{(x,s)} = \tilde{\eta}_{(x,s)} \circ p_{(x,s)}, \qquad p_{(x,0)} = \mathrm{Id}$$

Now we can apply the flow version of the Livšic theorem to conclude that there exists $\tilde{\phi} \in C^\alpha\big(\tilde{M}, \mathrm{Diff}^{r-1}(N)\big)$ such that

$$\tilde{\Phi}\big((x, s), t\big) = \tilde{\phi}(\tilde{f}^t(x, s) \circ \tilde{\phi}(x, s).$$

If we take $s = 0$ and $t = n$ we obtain

$$\tilde{\Phi}\big((x, 0), n\big) = \tilde{\phi}(f^n x, 0) \circ \tilde{\phi}(x, 0).$$

However we know that $\tilde{\Phi}\big((x,0),n\big) = \Phi(x,n)$ by construction and hence defining $\phi(x) = \tilde{\phi}(x,0)$ we obtain a solution to the coboundary equation. $\square$

6. Existence of invariant conformal structures on the stable and unstable bundles

In this section we will consider possibility of defining metrics on the stable and unstable bundles of Anosov systems that make the mapping conformal.

Of course, the existence of expanding and contracting directions in an Anosov map, makes it impossible to have metrics defined on the whole tangent bundle which make the map conformal. The conformal structures we consider in this section correspond to sub-Riemannian metrics on the manifold, not to Riemannian ones. In order to be able to do analysis on the manifold, we will assume that the manifold is equipped with a Riemannian metric, which we will assume analytic and which we will refer to as *background metric*.

Nevertheless, the existence of conformal metrics on the stable and unstable bundles is a useful tool in the study of rigidity questions. In [dlL02, dlL04b] it was shown that for conformal Anosov systems, the only obstructions to smooth conjugacy were the eigenvalues at periodic orbits (the paper above included an extra assumption about the existence of global frames of reference in the manifold, which we now remove). One motivation for the papers above was to understand geometrically the paper [CM97], which studied related problems for analytic families on the torus. The papers [KS03, Sad05] went further in the study of geometric properties and showed that conformal Anosov systems are smoothly equivalent to algebraic ones. Particularly interesting systems of conformal Anosov systems are the geodesic flows on some manifolds [Yue96]. For these systems, the results mentioned above give a very strong rigidity of the manifolds.

Of course, conformal metrics have played an important role in the theory of rigidity of manifolds. Thus, the study in this section provides a link between the dynamical rigidity of Anosov systems and the geometric rigidity [Mos68, Mos73]. Indeed, once the existence of a metric on the stable bundle that makes the mapping conformal is established, the arguments of [dlL02] are very similar to those of [Mos68, Mos73]. Indeed [Jen02] gave a proof of some particular cases of the results in [Mos68, Mos73] using methods from the theory of differentiable rigidity.

The main result of this section will be Theorem 6.8 which gives necessary and sufficient conditions conditions for the existence of families

of conformal structures on the stable and unstable bundles. The conditions involve the spectrum of the derivative of the return maps at periodic orbits.

The proof of Theorem 6.8 that we will present is remarkably similar to our proof of Theorem 3.1. We leave to the reader the task of formulating the corresponding result on existence of conformal metrics invariant under Anosov flows. The proof follows along extremely similar lines. We also note that similar arguments can be used in the study of other geometric structures.

Putting together Theorem 6.8 and the global results of [KS03, Sad05], we obtain that the global structure of the manifold is determined by the eigenvalues at periodic orbits. In the case of geodesic flows, it is well known that the eigenvalues at periodic orbits are determined by the spectrum of the Laplacian [GK80a] (provided that the length spectrum is simple).

We also note that in [dlLS05], one can find the result that if there is an invariant conformal structure that is in L^p for p sufficiently large, then there is a smooth conformal structure.

6.1. **Definitions and some elementary results.** We start by briefly reviewing the theory of quasi-conformal maps and setting the notation. All the results in this section are rather standard. Sources that we have found useful are [GP] and [Väi71].

Define the distortion of a differentiable map f at a point x with respect to a metric g, denoted $K_g(f, x)$, by

$$K_g(f,x) := \frac{\max_{\substack{|v|=1 \\ v\in T_xM}} |Df(x)v|_g}{\min_{\substack{|v|=1 \\ v\in T_xM}} |Df(x)v|_g} \tag{43}$$

or, equivalently,

$$K_g(f,x) := \|Df(x)\| \cdot \|Df^{-1}\bigl(f(x)\bigr)\| \tag{44}$$

Of course $K_g(f, x) \geq 1$. We say that the map is conformal with respect to g if $K_g(f, x) = 1$ for all x. Note that (44) makes it clear that

$$K_g(f,x) = K_g(f^{-1}, f(x)), \tag{45}$$

and in particular, f is conformal if and only if f^{-1} is conformal.

If we take the sup and inf in (43) when v ranges over a sub-bundle E of TM, we obtain the distortion along the sub-bundle E, which we will denote by $K_{g,E}(f, x)$.

Since any two metrics in a compact manifold are equivalent we have for some constant $C_{g,\tilde{g}} > 0$

$$C_{g,\tilde{g}}^{-1} K_{\tilde{g},E}(f,x) \le K_{g,E}(f,x) \le C_{g,\tilde{g}} K_{\tilde{g},E}(f,x) \tag{46}$$

where $C_{g,\tilde{g}}$ depends on the metrics g and $\tilde{g}$ but not on the map f or the sub-bundle E.

The distortion and distortion along bundles satisfy a sub-cocycle property

$$K_{g,E}(f_1 \circ f_2, x) \le K_{g,Df_2E}(f_1, f_2(x))\, K_{g,E}(f_2, x). \tag{47}$$

This follows from the chain rule and sub-multiplicativity of the operator norm.

We define $K_{g,E}(f) = \max_{x \in M} K_{g,E}(f,x)$.

In particular, when E is an invariant sub-bundle $Df(x)E_x \subset E_{f(x)}$ we have:

$$K_{g,E}(f^n) \le [K_{g,E}(f)]^n. \tag{48}$$

Hence, taking logarithms in (47) and using an elementary sub-additive argument

$$\overline{K}_E(f) = \lim_{n \to \infty} \big(K_{g,E}(f)\big)^{1/n}$$

exists. By (46), $\overline{K}_E(f)$ is independent of the metric. It can be seen, but we will not use it here, that the distortion along a bundle is closely related to the spectral properties of the weighted shift operator along the bundle.

We also recall the following easy results about distortions of linear operators on a fixed metric space, which we will use later to study the derivatives at fixed points.

Since $\|Av\|^2 = \langle v, A^*Av \rangle$, we have $\|A\|^2 = \max \operatorname{spec}(A^*A)$, $\|A^{-1}\|^{-1} = \min \operatorname{spec}(A^*A)$. Hence

$$K(A)^2 = \frac{\max \operatorname{spec}(A^*A)}{\min \operatorname{spec}(A^*A)} \ .$$

Proposition 6.1. *If $K(A) = 1$, then $\hat{A} = \dfrac{1}{\det(A)^{1/n}} A \in O(n)$, the orthogonal group corresponding to the metric.*

Proof. Since $K(A) = 1$ there is a constant $C > 0$ such that $\|Av\| = C\|v\|$ for all $v \in \mathbb{R}^n$. Since $\det \hat{A} = 1$ we must have $C = 1$. The desired result follows from the polarization argument. □

Proposition 6.2.

$$\|A\| \le |\det(A)|^{1/n} K(A) \tag{49}$$

Proof. The desired result (49) is obvious for diagonal operators and, hence for diagonalizable, in particular symmetric operators.

To prove the general case, we note that

$$\|A\|^2 = \|AA^*\| \le \det(AA^*)^{1/n} K(AA^*) \le \det(A)^{2/n} K(A)^2$$

Note also that the result is obvious for positive definite symmetric matrices since all the quantities in the formula can can be expressed in terms of the eigenvalues.

□

6.2. **Results and their proofs.** In this subsection we formulate and prove Theorem 6.3 which shows that the conformal properties of a transitive Anosov system are determined by the behavior at periodic orbits. Theorem 6.8 shows that, some spectral conditions on the periodic orbits are enough to obtain the existence of invariant conformal structures.

Similar theorems were proved in [dlL02] under some extra hypothesis about the manifold, in particular the existence of some trivialization of the bundle. The proof presented here is much more geometric. Indeed, it is remarkably similar to the proof of Theorem 3.1. We will consider the propagation of the structure and we will use the properties of the map to show that the behavior at periodic orbits controls what happens on a dense orbit. In some auxiliary lemmas, to obtain the equivalence of the hypothesis on the behavior at periodic orbits with other hypothesis, we will need to use the specification property of transitive Anosov systems.

Theorem 6.3. *Let M be a compact Riemannian manifold. Let f be a $C^{1+\alpha}$ $(0 < \alpha \le \mathrm{Lip})$ topologically transitive λ-hyperbolic Anosov diffeomorphism on M. Let E be a sub-bundle of the stable bundle E^s, which is invariant under f.*

Assume:

i) *There exists a constant C_{per} such that, whenever $f^N(x) = x$, with N the minimal period of x, we have*

$$K_{g,E}(f^N, x) \le C_{\mathrm{per}} \ .$$

ii) *$K_{g,E}(f) \le \rho$ with $\rho\,\lambda^\alpha < 1$.*

Then, there exists $C > 0$ such that for all $n \in \mathbb{N}$, we have

$$K_{g,E}(f^n) \le C.$$

Of course, an identical result holds for the unstable bundle.

Remark 6.4. *There is a version of Theorem 6.3 for flows. We leave the straightforward formulation as well as the proof to the reader.*

The proof of the result for flows requires only minor modifications of the proof we present here. The required modifications can be read off the corresponding modifications made to the proof of Theorem 3.1 to get the proof of Theorem 3.4. The only difference lies in the version of Anosov Closing Lemma that we use and the fact that for flows we have a term corresponding to the change in period to control.

Remark 6.5. *Note that because the distortion for a metric is equivalent to the distortion for another (see* (46)*) if the hypothesis i) or the conclusions hold for one metric they hold for any other metric. Hypothesis ii) on the other hand, depends on the metric g. Later, when it is better motivated by the proof, we will introduce a replacement hypothesis* (57)*, which, though harder to state, is more geometrically natural. We note that this hypothesis is a close analogue of the localization estimates of the first part of this paper, since it can be interpreted as a spectral condition for a transfer operator.*

Proof. We will show that for all $n \in \mathbb{N}$, we have $K_{g,E^s}(f^n) \le C$. We recall that by the specification property of transitive Anosov systems [KH95, Theorem 18.3.12], given $\epsilon > 0$ sufficiently small we can find $L \in \mathbb{N}$ such that for every $x \in M$ and $n \in \mathbb{N}$, there exists a period point p with minimal period $n + m$ with $0 \le m \le L$ that satisfies

$$d_M\big(f^i(x), f^i(p)\big) \le \epsilon \qquad 0 \le i \le n\ .$$

We will choose one such $\epsilon > 0$ that will remain fixed for the rest of the proof. This ϵ will have to satisfy a finite number of smallness conditions which we will make explicit when we need them.

By the local product structure [KH95, Proposition 6.4.21] we have

$$W^s_{\text{loc}}(p) \cap W^u_{\text{loc}}(x) = \{z\}$$

with $d(x, z) < \epsilon$ and $d(p, z) < \epsilon$. We may suppose that

$$d_M\big(f^i z, f^i x\big) \le \epsilon \qquad 0 \le i \le n.$$

We note that

$$\begin{aligned} K_{g,E^s}(f^n, p) &\le K_{g,E^s}(f^{n+m}, p)\, K_{g,E^s}(f^{-m}, p) \\ &\le C_{\text{per}}\, K_{g,E^s}(f^{-m}) \end{aligned} \tag{50}$$

with C_{per} the constant in assumption i). Hence, it suffices to argue that it is possible to control $K_{g,E^s}(f^n, x)$ in terms of $K_{g,E^s}(f^n, p)$.

6.2.1. *Some local coordinates.* We will find it convenient to use matrix notation so we introduce a coordinate patch about each point of the orbit of x. It is important to note that these coordinate patches do not need to agree in the regions where they overlap. Hence, they do not impose any restriction on the manifold M. Similar constructions happen in [HPPS70].

One convenient way of choosing these coordinate systems is picking a coordinate system ψ_i on $T_{f^i(x)}M$ with

$$\langle u, v \rangle_{g(f^i(x))} = \langle \psi_i u, \psi_i v \rangle_2$$

and then setting

$$\Psi_i(y) = \psi_i \circ \exp^{-1}_{f^i(x)}(y)$$

where the domain U_i chosen as balls of radius $1/2$ the injectivity radius of the metric. We neither assume, nor require, that the U_i are disjoint. Notice, however that these coordinate patches, centered around each point in the orbit include balls of radius bounded uniformly from below. Furthermore, the coordinate functions are C^r diffeomorphisms and they are uniformly C^r. So that, to show that a geometric object is C^r it suffices to show that its coordinates representations are uniformly C^r. Furthermore, the C^r norm of a geometric object will be equivalent to the supremum of the C^r norms of the coordinate representations.

Once we choose a system of coordinates, we can identify $Df(y)$, for $y \in U_i \cap f^{-1}(U_{i+1})$, with the matrix

$$\eta_i(y) = D\Psi_{i+1}(f(y))\, Df(y)\, (D\Psi_i(y))^{-1}. \tag{51}$$

We have chosen the notation η_i by analogy with the proof of Theorem 3.1.

We can now proceed in a way very similar to the proof of Theorem 3.1. For $y \in U_0$ we define

$$\Phi(y, m) = \eta_{m-1}(f^{m-1}y) \cdots \eta_0(y), \qquad m \geq 1$$

and for $y \in U_n$ we define

$$\Phi(y, -m) = \eta_{n-m}(f^{-m}y) \cdots \eta_{n-1}(f^{-1}y), \qquad m \geq 1$$

where these products are defined. Our goal is to show $K\big(\Phi(x, n)\big) < C$. By compactness, and our choice of geometrically natural coordinate systems, showing that the distortion of the coordinate representation of the derivative cocycle is uniformly bounded suffices to show that the distortion of the derivative cocycle itself is uniformly bounded.

If y, z are points whose orbits are converging in forward time so that $f^i(z)$ is always in the coordinate neighborhood of $f^i(y)$, we can use the same coordinate patch.

Define $C_F(m) = \Phi^{-1}(p,m)\Phi(z,m)$

Remark 6.6. *The point of introducing coordinates is to avoid unnecessary complication with connections. Intrinsically* $C_F(m) : T_zM \to T_pM$ *defined by* $Df^{-m}(f^m(p))SDf^m(z)$ *where* S *is an identification between* $T_{f^m(z)}M$ *and* $T_{f^m(p)}M$. *Note that, because the orbits of* z *and* p *are converging, we can always define the comparisons. Intuitively, as the points converge, the identifications become less important. Using the identifications between the end points is a possible alternative setup. This is what are called* connectors *in* [HPPS70].

We have the recurrence:

$$\begin{aligned} C_F(m+1) &= \Phi^{-1}(p,m)\eta^{-1}(f^mp)\eta(f^mz)\Phi(z,m) \\ &= \Phi^{-1}(p,m)\big[\eta^{-1}(f^mp)\eta(f^mz) - \mathrm{Id}\big]\Phi(z,m) \\ &\quad + C_F(m) \end{aligned} \tag{52}$$

Of course, we are interested only in the distortion of $C_F(m)$, so, we normalize the matrices to have determinant 1 with respect to the background metric.

$$\begin{aligned} \hat\eta_i(y) &= \frac{1}{(\det \eta_i(y))^{\frac{1}{n}}}\eta_i(y) \\ V_i(y) &= (\det \eta_i(y))^{\frac{1}{n}}. \end{aligned}$$

For $\epsilon > 0$ sufficiently small we can choose $\tilde\rho > 0$ and $\sigma > 0$ such that $\sigma\tilde\rho^2\lambda^\alpha < 1$ and

$$\begin{aligned} K\big(\eta_i(y)\big) &\le \tilde\rho \qquad \text{for } d_M(y, f^ix) < \epsilon, \\ \frac{V_i(y_1)}{V_i(y_2)} &\le \sigma \qquad \text{for } d_M(y_1, f^ix) < \epsilon \text{ and } d_M(y_2, f^ix) < \epsilon. \end{aligned}$$

The recurrence (52) can be written as:

$$\begin{aligned} C_F(m+1) = {}& \frac{V_m(f^mz)}{V_m(f^mp)} \cdots \frac{V_0(z)}{V_0(p)} \\ & \cdot \hat\Phi^{-1}(p,m)\big(\hat\eta^{-1}(f^mp)\hat\eta(f^mz) - \mathrm{Id}\big)\hat\Phi(z,m) + C_F(m). \end{aligned} \tag{53}$$

Since $f \in C^{1+\alpha}$ we have $\hat\eta_i \in C^\alpha$. Thus, since $z \in W^s(p)$

$$\|\hat\eta^{-1}(f^mp)\hat\eta(f^mz) - \mathrm{Id}\,\| \le C_2(\lambda^\alpha)^m.$$

Thus, by Proposition 6.2,

$$\|C_F(m+1) - C_F(m)\| < C_2\big(\sigma\tilde\rho^2\lambda^\alpha\big)^m.$$

Thus $\|C_F(m)\|$ is uniformly bounded. Similarly we obtain $\|C_F^{-1}(m)\|$ is uniformly bounded. Thus

$$\begin{aligned} \|\Phi(z,n)\| &\le \|\Phi(p,n)\|\,\|C_F(n)\|, \\ \|\Phi^{-1}(z,n)\| &\le \|\Phi^{-1}(p,n)\|\,\|C_F^{-1}(n)\|. \end{aligned} \tag{54}$$

Now we perform the same computations along orbits converging exponentially in backwards time. Let $C_R(m) = \Phi^{-1}(f^n x, -m)\Phi(f^n z, -m)$ and perform the same computations to obtain $\|C_R(m)\|$ and $\|C_R^{-1}(m)\|$ are uniformly bounded. Finally observe that we have the pseudo-cocycle property

$$\begin{aligned} \Phi^{-1}(f^n x, -n) &= \Phi(x,n) \\ \Phi(f^n z, -n) &= \Phi^{-1}(z,n) \end{aligned}$$

and so

$$\begin{aligned} \|\Phi(x,n)\| &\le \|C_R(n)\|\,\|\Phi(z,n)\| \\ \|\Phi^{-1}(x,n)\| &\le \|\Phi^{-1}(z,n)\|\,\|C_R^{-1}(n)\|. \end{aligned} \tag{55}$$

Replacing the norms $\|\Phi(z,n)\|$ and $\|\Phi^{-1}(z,n)\|$ in (55) with their estimates from (54) and taking the product we obtain

$$K\big(\Phi(x,n)\big) \le K\big(C_F(n)\big)\, K\big(C_R(n)\big)\, K\big(\Phi(p,n)\big). \tag{56}$$

We have shown that $K\big(C_F(n)\big)$ and $K\big(C_R(n)\big)$ are uniformly bounded. By our remarks at the outset we have $K\big(\Phi(p,n)\big)$ is uniformly bounded due to assumption i). $\square$

Remark 6.7. *We call attention to the fact that the only place in the proof of Theorem 6.3 where we use the hypothesis ii) is in the estimates of* (53)*. What we actually need is that the norms of* Δ_F^m *and* Δ_R^m *defined by*

$$\begin{aligned} \Delta_F^m(A) &:= \hat{\Phi}^{-1}(p,m)\, A\, \hat{\Phi}(z,m), \\ \Delta_R^m(A) &:= \hat{\Phi}^{-1}(f^n x, -m)\, A\, \hat{\Phi}(f^n z, -m) \end{aligned}$$

satisfy

$$\|\Delta_R^m\|, \|\Delta_F^m\| \le C\,(\sigma\lambda^\alpha)^{-m} \tag{57}$$

for n *large enough.*

Note that the condition (57) *on the asymptotic growth of the of cocycles is independent of the background metric. It can be used in place of assumption ii) in Theorem 6.3. Note the analogy with the localization estimates in the first part of this paper. In subsequent results (Theorem 6.8) we will also have similar hypothesis.*

Now we come to the second main result in this section, which characterizes the existence of conformal structures by the behavior at periodic orbits.

Theorem 6.8. *Let M be a compact d-dimensional Riemannian manifold endowed with a Riemanian metric g. (We refer to such a metric as the background metric.) Let f be a $C^{1+\delta}$ topologically transitive Anosov diffeomorphism.* $(0 < \delta \leq \mathrm{Lip})$.

i) *Assume whenever $f^N(x) = x$, then*
$$Df^N|_{E^s} = \gamma_{S,N}(x)\,\mathrm{Id}$$
for some real numbers $\gamma_{S,N}(x)$

ii) *Assume that*
$$K_{g,E^s}(f) \leq a$$
with a sufficiently close to 1.

Then there exists a C^δ metric g^s on E^s such that f is conformal on the stable leaves with respect to g^s.

Analogous results hold also for unstable bundles and for Anosov flows.

Of course the metrics are highly non-unique since we can multiply by an arbitrary function.

Remark 6.9. *Note that hypothesis i) does not depend on the background metric but hypothesis ii) does.*

As mentioned before, later we will formulate a geometrically natural – but somewhat harder to state – hypothesis, (58) *which can be used in place of ii).*

Remark 6.10. *Note that if we fix the conformal structure at one point x^*, the fact that f is conformal, determines it at fx^*. If we choose x^* so that its orbit is dense, the conformal structure at x^* determines it in the whole manifold. Hence, there is at most a finite dimensional family of invariant conformal structures. The proof of Theorem 6.8 is done by choosing a conformal structure at x^*, propagating the conformal structure along the dense orbit of x^*, and then, using the hypothesis on the spectrum of the periodic orbits showing it extends to the whole manifold.*

Note that then, we prove that, under the hypothesis on periodic orbits, we get that there is a family of invariant conformal structures with the dimension of the space of conformal structures at one point. Conversely, if there is family of conformal structures invariant under the map whose dimension is the dimension of conformal structures at one point, then the derivative at a fixed point has to be the a multiple of the identity.

Remark 6.11. *A result very similar to Theorem 6.8 was proved in* [dlL02] *but the proof required the existence of a global frame in the manifold. It was shown in* [dlL02] *that if the map f is C^r, continuous invariant conformal structures are actually $C^{r-1-\epsilon}$, $r \in \mathbb{N} \cup \{\infty, \omega\}$. The proof of the bootstrap of regularity in* [dlL02] *is very geometric and works without extra assumptions on the existence of frames. So, we will just refer to that paper. In* [dlLS05]*, it was shown that if an invariant conformal structures is in L^p for p large enough, then it is continuous and, therefore differentiable.*

The papers [KS03, Sad05, KS07] *show that the existence of a conformal structure on the stable and the unstable foliations for an Anosov systems, implies also some global properties of the manifold.*

Proof. Let x^* be a point with a dense orbit. We will define the desired metric along the orbit of x^* and show it extends to the whole manifold in a Hölder fashion.

We consider the bundle isomorphism $f_\#$ on the bundle of quadratic forms on E^s. Denoting the space of quadratic forms on E^s_x by Q_x we define $f_\# : Q_x \to Q_{f(x)}$ by

$$f_\# g = (\det Df_s(x))^{2/d} g(Df_s^{-1}(f(x)))^{\otimes 2}$$

where Df_s denotes the derivative of f restricted to E^s and d is the dimension of the stable bundle. The determinant is measured with respect to the background metric g.

We note that we can use the background metric g to measure the norm of $f_\#$.

We claim that, by assuming that $K_{g,E^s}(f)$ is sufficiently close to one, we can ensure that $\|f_\#\|$ is as close to 1 as we want. Hence, using assumption ii) we can assume in the proof that $\|f_\#\|$ is sufficiently close to 1.

Indeed, if we choose coordinates in E^s_x, $E^s_{f(x)}$ in such a way that g_x, $g_{f(x)}$ are represented by the identity matrix, the operator $f_\#$ reduces to the operator

$$\mathcal{L}(S) = A^t S A / \det(A)^{2/n}$$

acting on the space of symmetric matrices, where A is the coordinate representation of $Df^{-1}(f(x))$.

Applying Proposition 6.2 we obtain $\|A/|\det A|^{1/d}\| \leq K(A)$ from which the claim follows.

The hypothesis that we will need in the rest of the argument is

$$\text{ii}') \qquad \|f_\#^\ell\| \leq C\mu^n - 2\ell \tag{58}$$

with μ smaller than (λ^δ) — with λ the hyperbolicity exponent.

The rest of the proof is very similar to the proof of Theorem 3.1 and Theorem 6.3. We pick a metric on $g^s_{x^*}$ on $E^s_{x^*}$ and define

$$g^s_{f^n x^*} := f^n_\# g^s_{x^*}.$$

To check that the metric g^s defined along the orbit of x^* extends in a Hölder fashion to the whole of M we recall that by the Anosov Closing Lemma, Lemma 2.6, there exists $\epsilon > 0$ such that if $d(f^n x^*, f^{n+N} x^*) \le \epsilon$, then there is a periodic point p with $f^N p = p$ such that

$$d(f^{n+i} x^*, f^i p) \le \epsilon$$

We will estimate $f^N_\#$ restricted to $Q_{f^n x^*}$ using that, by hypothesis i), $f^N_\#$ restricted to Q_p . The estimates will depend only on ϵ but will be uniform in N.

From the Anosov Closing Lemma, Lemma 2.6 we obtain a "messenger" point $z \in W^u_{\mathrm{loc}}\big(f^n(x^*)\big) \cap W^s_{\mathrm{loc}}(p)$. We can take local coordinate systems defined on neighborhoods U_i around $f^{n+i}(x^*)$ for $0 \le i \le N$, in such a way that $f^i(p)$ is contained in the coordinate patch U_i.

We realize that $f_\#$ is C^α in the whole manifold. Denote by $\eta_i(y)$ the coordinate representation of $f_\#$ acting on Q_y for $y \in U_i$. Showing that the metric $f^{n+N}_\# g_{x^*}$ on $E^s_{f^{n+N} x^*}$ is close enough to the metric $f^n_\# g_{x^*}$ on $E^s_{f^n x^*}$ reduces to estimating

$$[\eta(f^N(q)) \cdots \eta(a)]^{-1} \eta(f^{n+N} x^*) \cdots \eta(f^n(x^*)) - \mathrm{Id}$$

As in the previous results we proceed to estimate

$$[\eta(f^N(z)) \cdots \eta(z)]^{-1} \eta(f^{n+N}(x^*)) \cdots \eta(f^n x^*) - \mathrm{Id}$$
$$[\eta(f^N(q)) \cdots \eta(q)]^{-1} \eta(f^N(z)) \cdots \eta(z) - \mathrm{Id}$$

The proof is exactly the same as in Theorem 3.1 and we refer to the proof of this result for the details. □

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